On the Theory and Application of Adomian Decomposition Method for Numerical Solution of Second-Order Ordinary Differential Equations.

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ABSTRACT

This paper present a new numerical method namely Adomian Decomposition Method (ADM) which is suitable for obtaining an approximate solution to second order ordinary differential equations. The method is applied to some examples and the results indicate that the method is reliable, accurate and converges rapidly.

(Keywords: Adomian decomposition method, approximate solution, convergence, second order differential equations)

INTRODUCTION

It is a well known and documented fact that many phenomena in engineering, the sciences, management, and economics can be modeled using the theory of derivatives and integrals. It is also interesting to note that solutions to most differential equations that arise from these models cannot be easily obtained by analytical means. Therefore, approximate solutions are needed which are generated by numerical techniques.

Some of the existing methods are based on discretization and they only allow the solutions to a given ordinary differential equations at a given interval. The above deficiency leads to a situation where some fundamental phenomena are easily avoided. The ADM is a relatively new approach, which provides an analytic approximation to linear and none linear problems. The method is quantitative rather than qualitative. It is analytic and it requires neither linearization nor perturbation. It is also continuous with no resort to discretization. The method provides the solution as an infinite series in which each term can be determined.

Throughout, we shall consider the equation of the form:

$$y^{II} = f(x, y), y(0)y_0, y^I(0) = y, x \in [0, b] \qquad \textbf{(1)}$$

We shall proceed to discuss the basic theory and concepts of Adomian Decomposition Method (ADM).

THE THEORY AND CONCEPTS OF ADOMIAN DECOMPOSITION METHOD (ADM)

The method consists of splitting the given equation into linear and non-linear parts, inverting the highest order derivative operator contained in the linear operator on both sides, identifying the initial conditions and the terms involving the independent variables alone as an initial approximation, decomposing the unknown function into a series whose components can be easily computed, decomposing the non-linear function in terms if polynomial called Adomian's polynomials, and finding the successive terms of the series solution by recurrent relation using the polynomials obtained (cf. Adomian 1988).

To solve problems of the form (1), we write it in an operator form as:

Ly= f(x, y) **(2)**

where the differential operator L is given as:

$$L = \frac{d^2}{dx^2}$$

The inverse operator:

$$L^{-1} = \int_0^x \int_0^x dx dx \qquad \textbf{(3)}$$

If we operate L^{-1} on both sides of (2) and impose the initial conditions we obtain:

$$y(x) = y_0 + y_i x + L^{-1}(f(x,y)) \quad \textbf{(4)}$$

The Adomian decomposition method introduces the solution $y(x)$ by an infinite series of components:

$$y(x) = \sum_{n=0}^{\infty} An \quad \textbf{(5)}$$

and the non-linear function f(x, y) by an infinite series of polynomials:

$$f(x,y) = \sum_{n=0}^{\infty} An \quad \textbf{(6)}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently, and the Adomian's Polynomial s A_n can be calculated for various classes of non-linearity according to algorithms recently set by G.Adomian and R. Rach (1992).

If we substitute (5) and (6) into (4), we obtain:

$$\sum_{n=0}^{\infty} y_n(x) = y_0 + y_1 x + L^{-1}[\sum_{n=0}^{\infty} An] \quad \textbf{(7)}$$

We next determine the components $y_n(x)$ for which n$\geq$0. We first identify the zero-th component $y_0(x)$ by all terms that arise from the initial conditions. The remaining components are determined by using the preceding component. Each term of the series (5) is given by the recurrent relation:

$$\left.\begin{array}{l} y_0(x) = y_0 + y_1 x \\ y_{n+1}(x) = L^{-1} A_n, n \geq 0 \end{array}\right\} \quad \textbf{(8)}$$

It must be stated here that all terms of series (5) cannot be computed and the solution of (1) will be approximated by series of the form:

$$\Phi_N(x) = \sum_{n=0}^{N-1} y_n(x) \quad \textbf{(9)}$$

The method reduces significantly the massive computation which may arise if discretization methods are used for the solution of non non-linear problems.

APPLICATIONS AND RESULTS

Example 1

Consider the linear equation:

$$y^{II} = x + y, \qquad y_{(0)} = 1, \qquad y^{I}(0), \qquad x \in [0,5] \quad \textbf{(10)}$$

with the theoretical solution:

$y(x) = e^x - x$.

We apply ADM operator to equation (10) to produce:

$$Ly = x + y \quad \textbf{(11)}$$

Operating L^{-1} on both sides of (11) and use the initial conditions, we obtain:

$$y(x) = 1 + L^{-1}(x) + L^{-1}y \quad \textbf{(12)}$$

By using (7), we obtain:

$$\sum_{n=0}^{\infty} y_n(x) = 1 + \int_0^x\int_0^x xdxdx + \int_0^x\int_0^x \sum_{n=0}^{\infty} y_n \quad \textbf{(13)}$$

The ADM introduces the recursive relation:

$$y_0(x) = 1 + L^{-1}(x) = 1 + \frac{x^3}{6}$$

$$y_{(n+1)} = L^{-1}(y_n) \qquad \text{n} \geq 0$$

We can then proceed to compute the first few terms of the series:

$$y_1(x) = L^{-1}(y_0) = \int_0^x \int_0^x \sum_{n=0}^{\infty} (1 + \frac{x^3}{6}) dx dx = \frac{x^2}{2} + \frac{x^5}{120}$$

$$y_2(x) = L^{-1}(y_1) = \int_0^x \int_0^x (\frac{x^2}{2} + \frac{x^5}{120}) dx dx = \frac{x^4}{24} + \frac{x^7}{5040}$$

$$y_3(x) = L^{-1}(y_2) = \int_0^x \int_0^x (\frac{x^4}{24} + \frac{x^7}{5040}) dx dx = \frac{x^6}{720} + \frac{x^7}{362880}$$

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$$y_n(x) = \frac{x^{2n}}{(2n)!} + \frac{x^{(3+2n)}}{(3+2n)!} \quad \textbf{(14)}$$

Hence,

$$\Phi_{10}(x) = \sum_{n=0}^{9} y_n(x), \qquad \mathrm{n} \geq 0 \quad \textbf{(15)}$$

For application purpose, only the first ten terms of the series is computed. Table (1) compares the result obtained using ADM with exact solution. It is obvious that the result is in agreement with the exact solution. Higher accuracy can be obtained by evaluating more components of the series (15).

Table 1

At H = 0.5

X	ADOMIAN	EXACT	ERROR
0.00	1.00000000	1.00000000	0.00000000
0.50	1.14872134	1.14872122	0.00000012
1.00	1.71828175	1.71828175	0.00000000
1.50	2.98168921	2.98168898	0.00000024
2.00	5.38905621	5.38905621	0.00000000
2.50	9.68249416	9.68249416	0.00000000
3.00	17.08553696	17.08553696	0.00000000
3.50	29.61545563	29.61545181	0.00000381
4.00	50.59815216	50.59814835	0.00000381
4.50	85.51713562	85.51712799	0.00000763
5.00	143.41316223	143.41316223	0.00000000

At H = 0.2

X	ADOMIAN	EXACT	ERROR
0.00	1.00000000	1.00000000	0.00000000
0.10	1.00517094	1.00517094	0.00000000
0.20	1.02140284	1.02140272	0.00000012
0.30	1.04985881	1.04985881	0.00000000
0.40	1.09182465	1.09182477	0.00000012
0.50	1.14872134	1.14872122	0.00000012
0.60	1.22211874	1.22211885	0.00000012
0.70	1.31375277	1.31375265	0.00000012
0.80	1.42554104	1.42554104	0.00000000
0.90	1.55960333	1.55960321	0.00000012
1.00	1.71828198	1.71828210	0.00000012

Example 2

Let us consider the equation;

$$x^{II}(t)+e^{-rt}=0,\ x(t)=0,\ x(0)=1,\ x^{I}(0)=0,\quad t\in[0,1] \qquad \textbf{(16)}$$

where x(t) is the displacement at time t and γ is a positive constant which represent the model for a spring-mass system. For γ =0, the equation reduces to:

$$x^{II}(t)+x(t)=0,\ x(0)=1,\ x^{I}(0)=0 \qquad \textbf{(17)}$$

The exact solution of (17) is cost t. In an operator form, (16) becomes:

Lx = -x **(18)**

Operating L^{-1} on both sides of (18) and using the initial conditions, we have:

$$x(t)=x(0)+x^{I}(0)t-L^{-1}x(t)$$
$$x(t)=1-L^{-1}x(t)$$
$$\left.\begin{array}{l} x_0(t)=1 \\ x_{n+1}(t)=L^{-1}x_n(t) \qquad n\geq 0 \end{array}\right\} \qquad \textbf{(19)}$$

$$x_1(t)=-L^{-1}(x_0(t))=\int_0^t\int_0^t 1dtdt=\frac{-t^2}{2}$$
$$x_2(t)=-L^{-1}(x_1(t))=\int_0^t\int_0^t (\frac{-t^2}{2})dtdt=\frac{t^4}{4!}$$
$$x_3(t)=-L^{-1}(x_2(t))=\int_0^t\int_0^t (\frac{t^4}{4!})dtdt=\frac{-t^6}{6!}$$

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$$x_n(t)=\frac{(-1)^n t^{2n}}{(2n)!}$$

Consequently,

$$\Phi_{10}(t)=\sum_{n=0}^{9}x_n(t) \qquad \textbf{(20)}$$

Here, only the first ten terms of the decomposition series were used in evaluating the approximate solution for Table (2). The efficiency of this approach can be drastically enhanced by computing further terms of the series.

CONCLUDING REMARKS

In this section, a simple proof of convergence of Adomian's technique is presented. The ADM introduces the solution $y(x)$ of (1) as:

$$\sum_{n=0}^{\infty}y_n(x)=y_{(0)}+L^{-1}\sum_{n=0}^{\infty}A_n \qquad \textbf{(21)}$$

Where A_n 's are polynomials in $y_0, y_1 ... y_n$ called determining the sequence:

$$S_n=y_0+y_1+y_2+y_3...y_n \qquad \textbf{(22)}$$

For the study of the numerical resolution of (1) Cherrault and Rach (1995) used fixed-point theorem.

Theorem (Cherrault and Rach 1995).

Let N be an operator from a Hilbert space H into H and Y be the exact solution of (1) then

$$\sum_{n=0}^{\infty}y_n(x)$$

which is obtained by ADM, converges to y when there exist $\alpha\in[0,1]$ such that

$$\| y_{n+1}\| \leq \alpha \| y_k \|, \forall k\in NU\{0\}$$

Table 2

H = 0.2

X	ADOMIAN	EXACT	ERROR
0.00	1.00000000	1.00000000	0.00000000
0.20	0.98006660	0.98006660	0.00000000
0.40	0.92106104	0.92106098	0.00000000
0.60	0.82533562	0.82533562	0.00000000
0.80	0.69670671	0.69670671	0.00000000
1.00	0.54030228	054030228	0.00000000
1.20	0.36235774	0.36235771	0.00000003
1.40	0.16996713	0.16996706	0.00000007
1.60	-0.02919978	-0.02919967	0.00000012
1.80	-0.22721598	-0.22720228	0.00001369
2.00	-0.4163340	-0.41614705	0.00018698

H = 0.1

X	ADOMIAN	EXACT	ERROR
0.00	1.00000000	1.00000000	0.00000000
0.10	0.99500418	0.99500418	0.00000000
0.20	0.95533645	0.98006660	0.00000000
0.40	0.92106104	0.95533651	0.00000006
0.50	0.87758261	0.92106098	0.00000006
0.60	0.82533562	0.87758255	0.00000006
0.70	0.76484221	0.82533562	0.00000000
0.80	0.69670671	0.69670665	0.00000006
0.90	0.62160987	0.62160987	0.00000000
1.00	0.54030216	0.54030222	0.00000006

Proof, we have:

$$\left.\begin{array}{l} S_0 = y_0 = 0 \\ S_1 = y_1 \\ S_2 = y_1 + y_2 \\ . \\ . \\ . \\ S_n = y_1 + y_2 + y_3 + \ldots + y_n \end{array}\right\} \quad \textbf{(23)}$$

And we show that, $\{S_n\}^{\infty}_{n=0}$ is a Cauchy sequence in Hilbert H for this purpose, consider:

$$\| S_{n+1} - S_n \| = \| y_{N-1} \| \le \alpha \| y_n \| \le \alpha^2 \| y_{n-1} \| \ldots \le \alpha^{n-1} \| y_0 \|$$

But for every $n, m \in N, n \ge m$ we have:

$$\begin{aligned} \| S_n - S_m \| &= \| (S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{m-1} - S_m) \| \\ &\le \| S_n - S_{n-1} \| + \| S_{n-1} - S_{n-2} \| + \ldots + \| S_{n-1} - S_m \\ &\le \alpha^n \| y_0 \| + \alpha^{n-1} \| y_0 \| + \ldots + \alpha^{m+1} \| y_0 \| \\ &\le (\alpha^{m+1} + \alpha^{m+2} + \ldots) \| y_0 \| = \frac{\alpha^{m+1}}{1-\alpha} \| y_0 \| \end{aligned}$$

hence,

$$\lim \| S_n - S_m \| = 0$$
$$n, m \to s + \infty$$

i.e $\{S_n\}_{=0}^{+\infty}$ is a Cauchy sequence in the Hilbert space H and it implies that there exist S, SEH such that $\lim_{x \to \infty} S_n = S$

i.e. $S = \sum_{n=0}^{\infty} y_n$

We have been able to present and apply the ADM to second order differential equations. We have presented and compared the numerical solution using ADM with the theoretical solution. From our findings, we observed that more accuracy can be obtained by accommodating more terms in our decomposition series. One of the advantages of ADM is that it generates solutions over infinite intervals.

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SUGGESTED CITATION

Ibijola, E.A. and B.J. Adegboyegun. 2008. "On the Theory and Application of Adomian Decomposition Method for Numerical Solutions of Second-Order Ordinary Differential Equations". *Pacific Journal of Science and Technology*. 9(2):357-362.

Pacific Journal of Science and Technology