Motion of Particles of Non-Zero Rest Masses Exterior to a Spherical Mass Distribution with a Time Dependent Potential Field.

Chifu E. Ndikilar, M.Sc.[1]**, Adam Usman, Ph.D.**[2]**, and Osita C. Meludu, Ph.D.**[2]

[1]Department of Physics, Gombe State University, PMB 127, Gombe, Nigeria.
[2]Department of Physics, Federal University of Technology, PMB 2076, Yola, Adamawa State, Nigeria.

*E-mail: aausman@yahoo.co.uk
omeludu@yahoo.co.uk

ABSTRACT

In this article, we extend Schwarzschild's solution to Einstein's gravitational field equations. The equations of motion are derived for particles of non-zero rest masses exterior to a spherical mass distribution whose potential field is time dependent. The angular equations of motion for this field are found to be the same as that of Schwarzschild's field. The time dependent field affects mainly the time and the radial equations of motion.

(Keywords: equations of motion, general relativity, time-dependent potential field, Schwarzschild's solution)

INTRODUCTION

The 1907 generalized equivalent hypothesis demands that equations of Physics should be generally covariant. The overriding effect is that physical laws should be valid for any choice of space-time coordinates [1]. The method in this work involves use of general tensor analysis with some simplification. The derivation of tensor equations involves the manipulation of anyone of the fourteen orthogonal curvilinear coordinates in nature to find solutions to physical problems [2].

In Schwarzschild's well known metric, the potential field is assumed to be static for the spherically homogenous mass distribution [3]. Here, we introduce a time-dependent potential field to derive equations of motion for particles of non-zero rest masses. The implications are that such a particle would have a velocity that is less than the velocity of light. Thus, the potential field naturally comes into action as a result of gravitational effect of an astrophysical mass distribution. In the following discursions, spherical geometry will be assumed for the astrophysical body.

THEORETICAL FORMULATION

We consider a uniformly distributed spherical body of radius R and total rest mass M. The general relativistic field equations in the exterior region of the body are tensorially given as [3]:

$$G_{\mu\nu} = 0 \quad \textbf{(1)}$$

where $G_{\mu\nu}$ is the Einstein tensor. As in usual notation, the Greek subscripts run from 0 to 3, with 0^{th} component representing the time coordinate, and the 1^{st}, 2^{nd} and 3^{rd} components denote the location in space.

The Schwarzschild's metric is the solution of Einstein's gravitation field equation exterior to a static homogenous spherical body [3] given by;

$$g_{00} = 1 + \frac{2}{c^2} f(r) \quad \textbf{(2a)}$$

$$g_{11} = -\left[1 + \frac{2}{c^2} f(r)\right]^{-1} \quad \textbf{(2b)}$$

$$g_{22} = -r^2 \quad \textbf{(2c)}$$

$$g_{33} = -r^2 \sin^2\theta \quad \textbf{(2d)}$$

$$g_{\mu\nu} = 0 \text{ ; otherwise} \quad \textbf{(2e)}$$

where c is the speed of light in vacuum, $f(r)$ is an arbitrary function of the radial coordinate so that its distribution possesses spherical symmetry.

From the condition that these metric components should reduce to the field of a point mass located at the origin [4] and contain Newton's equations of motion in the gravitational field of the static homogenous spherical body, it follows that *f(r)* is the Newtonian gravitational scalar potential in the exterior region of the body defined in the field as:

$$f(r) = -\frac{GM}{r} \qquad \textbf{(3)}$$

where *G* is the universal gravitational constant.

It is well known that the solar system is not static [5]. That is, the gravitational scalar potential will not remain the same as *f(r)*. It will be a time dependent function, *f(t,r)* in the gravitational field. It is therefore justified to replace *f(r)* by *f(t,r)* in the metric (2a) to (2e) to yield:

$$g_{00} = 1 + \frac{2}{c^2} f(t,r) \qquad \textbf{(4a)}$$

$$g_{11} = -\left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \qquad \textbf{(4b)}$$

$$g_{22} = -r^2 \qquad \textbf{(4c)}$$

$$g_{33} = -r^2 \sin^2\theta \qquad \textbf{(4d)}$$

$$g_{\mu\nu} = 0 \text{; otherwise} \qquad \textbf{(4e)}$$

Equations (4a) to (4e) constitute the covariant metric tensor for time varying potential field of the spherical mass distribution. The consequent gravitational field affects the non-zero rest mass located at the exterior of the body.

To obtain the corresponding contravariant metric tensor, we apply the Quotient Theorem [6] of tensor analysis.

$$g^{\mu\sigma} g_{\sigma\nu} = \delta^{\mu}_{\nu} \qquad \textbf{(5)}$$

where δ^{μ}_{ν} is the Kronecker delta tensor defined thus:

$$\delta^{\mu}_{\nu} = \begin{cases} 0, & \mu \neq \nu \\ 1, & \mu = \nu \end{cases} \qquad \textbf{(6)}$$

By the Quotient Theorem, the contravariant metric tensor components are:

$$g^{00} = \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \qquad \textbf{(7a)}$$

$$g^{11} = -\left[1 + \frac{2}{c^2} f(t,r)\right] \qquad \textbf{(7b)}$$

$$g^{22} = -r^{-2} \qquad \textbf{(7c)}$$

$$g^{33} = -\left(r^2 \sin^2\theta\right)^{-1} \qquad \textbf{(7d)}$$

$$g^{\mu\nu} = 0 \text{; otherwise} \qquad \textbf{(7e)}$$

The coefficients of affine connection, defined by the metric tensor of space-time are found to be:

$$\Gamma^0_{00} = \frac{1}{2} g^{00} g_{00,0} \qquad \textbf{(8a)}$$

$$\Gamma^0_{01} \equiv \Gamma^0_{10} = \frac{1}{2} g^{00} g_{00,1} \qquad \textbf{(8b)}$$

$$\Gamma^0_{11} = -\frac{1}{2} g^{00} g_{11,0} \qquad \textbf{(8c)}$$

$$\Gamma^1_{00} = -\frac{1}{2} g^{11} g_{00,1} \qquad \textbf{(8d)}$$

$$\Gamma^1_{01} \equiv \Gamma^1_{10} = \frac{1}{2} g^{11} g_{11,0} \qquad \textbf{(8e)}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} g_{11,1} \qquad \textbf{(8f)}$$

$$\Gamma^1_{22} = -\frac{1}{2} g^{11} g_{22,1} \qquad \textbf{(8g)}$$

$$\Gamma^1_{33} = -\frac{1}{2} g^{11} g_{33,1} \qquad \textbf{(8h)}$$

$$\Gamma^2_{12} \equiv \Gamma^2_{21} = \frac{1}{2} g^{22} g_{22,1} \qquad \textbf{(8i)}$$

$$\Gamma_{33}^{2} = -\frac{1}{2} g^{22} g_{33,2} \quad \textbf{(8j)}$$

$$\Gamma_{13}^{3} \equiv \Gamma_{31}^{3} = \frac{1}{2} g^{33} g_{33,1} \quad \textbf{(8k)}$$

$$\Gamma_{23}^{3} \equiv \Gamma_{32}^{3} = \frac{1}{2} g^{33} g_{33,2} \quad \textbf{(8l)}$$

$\Gamma_{\mu\nu}^{\alpha} = 0$; otherwise **(8m)**

where the comma denotes partial differentiation with respect to (0,1,2) ≡ (ct, r, θ), and α ≡ (0, 1, 2, 3).

Equations (8a) to (8m) are next written in terms of (ct, r, θ, ϕ) given in the following equations

$$\Gamma_{00}^{0} = \frac{1}{c^3}\left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \frac{\partial f(t,r)}{\partial t} \quad \textbf{(9a)}$$

$$\Gamma_{01}^{0} \equiv \Gamma_{10}^{0} = \frac{1}{c^2}\left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \frac{\partial f(t,r)}{\partial r} \quad \textbf{(9b)}$$

$$\Gamma_{11}^{0} = -\frac{1}{c^3}\left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \frac{\partial f(t,r)}{\partial t} \quad \textbf{(9c)}$$

$$\Gamma_{00}^{1} = \frac{1}{c^2}\left[1 + \frac{2}{c^2} f(t,r)\right] \frac{\partial f(t,r)}{\partial r} \quad \textbf{(9d)}$$

$$\Gamma_{01}^{1} \equiv \Gamma_{10}^{1} = \frac{1}{c^2}\left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \frac{\partial f(t,r)}{\partial r} \quad \textbf{(9e)}$$

$$\Gamma_{11}^{1} = -\frac{1}{c^2}\left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \frac{\partial f(t,r)}{\partial t} \quad \textbf{(9f)}$$

$$\Gamma_{22}^{1} = -r\left[1 + \frac{2}{c^2} f(t,r)\right] \quad \textbf{(9g)}$$

$$\Gamma_{33}^{1} = -r^2 \sin\theta \left[1 + \frac{2}{c^2} f(t,r)\right] \quad \textbf{(9h)}$$

$$\Gamma_{12}^{2} \equiv \Gamma_{21}^{2} = r^{-1} \quad \textbf{(9i)}$$

$$\Gamma_{33}^{2} = -\frac{1}{2}\sin 2\theta \quad \textbf{(9j)}$$

$$\Gamma_{13}^{3} \equiv \Gamma_{31}^{3} = -r^{-1} \quad \textbf{(9k)}$$

$$\Gamma_{23}^{3} \equiv \Gamma_{32}^{3} = \cot\theta \quad \textbf{(9l)}$$

$\Gamma_{\mu\nu}^{\alpha} = 0$; otherwise **(9m)**

All equations so far are the tools for developing the equations of motion in the next section. To do so would require the use of the well known general relativistic equation of motion for particles of non-zero rest masses in a gravitational field [7]. The equation is given by:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^{\mu}\left(\frac{dx^\nu}{d\tau}\right)\left(\frac{dx^\lambda}{d\tau}\right) = 0 \quad \textbf{(10)}$$

where τ is the proper time.

FORMULATION OF EQUATIONS OF MOTION

We begin by setting μ = 0 in Equation (10). This leads to the time equation of motion as

$$\frac{d^2 x^0}{d\tau^2} + \Gamma_{\nu\lambda}^{0}\left(\frac{dx^\nu}{d\tau}\right)\left(\frac{dx^\lambda}{d\tau}\right) = 0 \quad \textbf{(11)}$$

Substituting Equations (9a) to (9c) into Equation (11) gives:

$$\frac{d^2x^0}{d\tau^2}+\Gamma^0_{00}\left(\frac{dx^0}{d\tau}\right)^2+2\Gamma^0_{01}\frac{dx^0}{d\tau}\frac{dx^1}{d\tau}+\Gamma^0_{11}\left(\frac{dx^1}{d\tau}\right)^2=0 \qquad \textbf{(12)}$$

Equation (12) can be written more explicitly as

$$\ddot{t}+\frac{1}{c^2}\left[1+\frac{2}{c^2}f(t,r)\right]^{-1}\frac{\partial f(t,r)}{\partial t}\dot{t}+\frac{2}{c^2}\left[1+\frac{2}{c^2}f(t,r)\right]^{-1}\frac{\partial f(t,r)}{\partial r}\dot{t}\dot{r}$$
$$-\frac{1}{c^4}\left[1+\frac{2}{c^2}f(t,r)\right]^{-1}\frac{\partial f(t,r)}{\partial t}\dot{r}^2=0 \qquad \textbf{(13)}$$

where the dot denotes differentiation with respect to the proper time, τ. Equation (13) is the time equation of motion for a particle of non-zero rest mass exterior to an astrophysical spherical distribution of mass of which potential field is time dependent. One application of Equation (13) is to deduce an expression for the variation of the time of a clock with the time dependent gravitational field.

We next set μ = 1 in Equation (10). This gives the radial equation of motion as:

$$\ddot{r}+\left[1+\frac{2}{c^2}f(t,r)\right]\frac{\partial f(t,r)}{\partial r}\dot{t}^2-\frac{1}{c^2}\left[1+\frac{2}{c^2}f(t,r)\right]^{-1}\frac{\partial f(t,r)}{\partial r}\dot{r}^2$$
$$+\frac{2}{c^2}\left[1+\frac{2}{c^2}f(t,r)\right]^{-1}\frac{\partial f(t,r)}{\partial t}\dot{t}\dot{r}-r\left[1+\frac{2}{c^2}f(t,r)\right]\dot{\theta}^2-r\sin^2\theta\left[1+\frac{2}{c^2}f(t,r)\right]\dot{\phi}^2=0 \qquad \textbf{(14)}$$

For pure radial motion, $\dot{\theta}=\dot{\phi}=0$. The radial Equation (14) therefore becomes:

$$\ddot{r}+\left[1+\frac{2}{c^2}f(t,r)\right]\frac{\partial f(t,r)}{\partial r}\dot{t}^2-\frac{1}{c^2}\left[1+\frac{2}{c^2}f(t,r)\right]^{-1}\frac{\partial f(t,r)}{\partial r}\dot{r}^2$$
$$+\frac{2}{c^2}\left[1+\frac{2}{c^2}f(t,r)\right]^{-1}\frac{\partial f(t,r)}{\partial t}\dot{t}\dot{r}=0 \qquad \textbf{(15)}$$

Equation (15) is the pure radial equation of motion for particles of non-zero rest masses located at exterior of the spherical mass distribution of time dependent potential field, *f(t,r)*. It is worth noting that Equation (15) reduces to Schwarzschild's pure radial equation of motion when *f(t,r)* ceases to be time dependent. By solving Equation (15) one would get an expression for the instantaneous speed of a particle of non-zero rest mass in the time dependent gravitational field.

Also, setting μ = 2 and μ = 3 respectively in Equation (10), the angular and azimuthal equations of motion are obtained for the particle of non-zero rest masses in the time dependent field as:

$$\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}-\frac{1}{2}\dot{\phi}^2\sin 2\theta=0 \qquad \textbf{(16)}$$

$$\ddot{\phi}+\frac{2}{r}\dot{r}\dot{\phi}+2\dot{\theta}\dot{\phi}\cot\theta=0 \qquad \textbf{(17)}$$

It should be noticed that Equations (16) and (17) are equal to angular and azimuthal equations of motion for a particle of non-zero rest mass in the Schwarzschild's field [5]. Therefore, the instantaneous azimuthal angular velocity from our field is exactly the same as that obtained from Newton's theory of gravitation [8] and Schwarzschild's metric [5].

CONCLUSION

The time, radial, angular, and azimuthal equations of motion are found respectively to be Equations (13), (14), (16), and (17) for particles of non-zero rest masses in the exterior of a spherical distribution of mass. The potential field of the mass distribution is time dependent. The physical implications of the results obtained in this work are:

1) The solution of the time equation of motion gives the variation of the time on a clock with the non-static gravitational field. That is, an expression for gravitational time dilation will result.

2) Solution of the radial equation of motion will yield an expression for the instantaneous speed of a particle of non-zero rest mass in the time dependent potential field.

3) The angular and azimuthal equations of motion are the same as those of obtained from the Schwarzschild's field and the Newtonian theory of gravitation. That is, the time varying field has no consequence on the angular and azimuthal equations of motion.

Finding solutions of the equations of motion are one task ahead. Attempts will further be made to use the coefficients of affine connection obtained for constructing Riemann-Christoffel, Ricci, and Einstein's tensor for the time varying field. It is expected that the efforts will yield the Einstein field equations for the time dependent gravitational field. However, the caveat is this: for physical significance, we will attempt to find a transformation for the time-varying potential field, *f(r,t)*, to satisfy the Birkhoff's theorem [3,9,10].

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ABOUT THE AUTHORS

Chifu E. Ndikilar, is currently an Assistant Lecturer in the Department of Physics, Gombe State University, Nigeria. He earned his B.Sc. (Physics) degree in 2002 from the University of Buea, Cameroon. He holds an M.Sc. (Physics), 2006, degree from the University of Jos, Nigeria. He is currently working towards his doctoral degree in General Relativity applied and applicable to gravitational phenomena.

Adam Usman is currently a Senior Lecturer in the Department of Physics, Federal University of Technology, Yola, (FUTY), Nigeria. His research interests include Quantum Optics, Nonlinear Optics, Relativistic Mechanics, and Applied Physics. He earned a B.Tech. (Physics) degree from FUTY in 1989, an M.Sc. (Solid State Physics), from University of Ibadan, in 1992, and a Doctorate degree in 2000 from the University of Science, Malaysia.

Osita C. Meludu is currently a Senior Lecturer in the Department of Physics, (FUTY), Nigeria. He holds a B.Sc. (Physics) degree, an M.Sc. (Geophysics) degree, and a Doctorate degree which he earned in 1986, 1989, and 1997, respectively. His research interests are in Applied Geophysics, Health Physics, and Relativistic Mechanics.

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