Periodic Solutions for a Non-Linear Boundary Value Problem (BVP) of a Fourth Order Differential Equation.

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ABSTRACT

Existence of periodic solutions for the Equation (14) subject to Equation (2) has been obtained via the generalized eigenvalue problem. The procedure consists essentially of first determining a set of appropriate conditions in respect of general constant coefficient ordinary differential equations of order four and then generalizing these conditions to suitable corresponding non-linear ordinary differential equations of order four. The method of research is by Leray-Schauder fixed point technique and the use of integrated and so-called energy equation is the mode for estimating the a priori bounds.

(Keywords: boundary value problems (BVP), a priori bounds, compact and equicontinuity, integrated and so-called energy equation)

INTRODUCTION

Consider the generalized eigenvalue problem:

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_4 x = -a_1\lambda\dot{x} \quad \textbf{(1)}$$

with boundary conditions

$$D^{(r)}x(0) = D^{(r)}x(2\pi),\ r = 0,1,2,3,\ D = \frac{d}{dt} \quad \textbf{(2)}$$

$a_1(\neq 0)$, a_2 and a_4 constants.

Let $x(t)$ be a solution of (1) – (2) for some λ and let $x'(t)$ have the Fourier expansion:

$x'(t) \sim \sum_{r=1}^{\infty}(d_r \cos rt + e_r \sin rt)$ for $t \in [0, 2\pi]$.

Split $x'(t)$ into two parts Y_1, Y_2 as follows:

$$Y_1 = \sum_{r\le\lambda}(d_r \cos rt + e_r \sin rt),$$
$$Y_2 = \sum_{r>\lambda}(d_r \cos rt + e_r \sin rt)$$

By multiplying (1) by $Y_1 - Y_2$ and integrating over $[0,\ 2\pi]$, it can be checked that

(I) Any $\lambda \neq m^2$ for each $m = 1, 2, \dots$ is not an eigenvalue of (1) – (2) for arbitrary a_2 if $a_4 \neq 0$

(II) Any $\lambda = m^2$ for some $m = 1, 2, \dots$ is an eigenvalue of (1) – (2) if and only if:

$$\chi(m) = m^4 - a_2 m^2 + a_4 \quad \textbf{(3)}$$

The statements (I) and (II) have important bearings on the solvability of the well known 2π periodic (BVP) for the non-autonomous equation:

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + \lambda a_1\dot{x} + a_4 x = p(t) \quad \textbf{(4)}$$

By (I) one expects existence of a 2π periodic solution for the equation:

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + g(t,\ \dot{x}) + a_4 x = p(t) \quad \textbf{(5)}$$

for arbitrary a_2, a_4 if g is such that $a_1^{-1}y^{-1}g(t,y)$ lies in suitable interval $\left[m^2,\ (m+1)^2\right]$, where m is a non-zero integer.

See Ezeilo (1997), Ezeilo and Omari (1989), Ezeilo and Onyia (1984).

Again (I) taken together with (II) leads one to expect solutions to the 2π periodic boundary value problem for the equation:

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = p(t) \qquad \textbf{(6)}$$

for arbitrary a_1 *and* a_3 if a_2 *and* a_4 satisfy

$$\chi(m) \neq 0 \text{ for } m = 1, 2, \ldots \qquad \textbf{(7)}$$

The Equation (7) is an improvement on previous results in Ezeilo and Tejumola (1979), which requires that $\chi(\lambda) \neq 0$ for all numbers λ.

Note that Equation (7) can also be established as a condition for existence of a 2π periodic solution of Equation (6) by going through the auxiliary equation for

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = 0.$$

That is without going into the generalized eigenvalue problem (1) – (2).

The objectives this paper are to use the hypotheses derived solely from Equation (7) as a basis for establishing the existence of a 2π periodic solution for Equation (6) in which $a_1, \ldots, a_4$ are all not necessarily constants and $p = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$ is bounded and 2π periodic in t uniformly in respect of other variables shown.

Since $\chi(m)$ can be written in the form

$$\chi(m) = \left(m^2 - \frac{1}{2}a_2\right)^2 + a_4 - \frac{1}{4}a_2^2 \qquad \textbf{(8)}$$

it is convenient in examining the implication of (7) to distinguish the following cases:

(i) $a_2 \leq 0$

(ii) $\frac{1}{2}a_2 = m^2$ for some integer $m > 0$

From (i), it is clear that $a_2 \leq 0$ by Equation (3)

$$\chi(m) > 0, \ \ if \ a_4 > 0. \qquad \textbf{(9)}$$

From (ii), $a_2 = 2N^2$ for some integer $N > 0$.

Here if $\inf_m \chi(m) = \chi(N) = a_4 - \frac{1}{4}(2N^2)^2 = a_4 - N^4$.

And so we can claim that $\chi(m) > 0$ provided that,

$$a_4 > N^4 = \frac{1}{4}a_2^2 \qquad \textbf{(10)}$$

The foregoing consideration of cases (i) and (ii) suggest that if p is sufficiently smooth, then a 2π periodic solution of the equation,

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \qquad \textbf{(11)}$$

exists for arbitrary a_1, a_3 if

$$a_2 \leq 0, \ a_4 > 0 \qquad \textbf{(12)}$$

or if

$a_2 = 2N^2$ for some integer $N > 0$ and $a^4 = N^4$. **(13)**

We now transfer these considerations to the more general equation

$$x^{(4)} + \psi(\ddot{x})\dddot{x} + h(x, \dot{x}, \ddot{x}, \dddot{x})\ddot{x} + \theta(\dot{x}) + \varphi(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \qquad \textbf{(14)}$$

where $\psi, h, \varphi, \theta, p$ depend on the argument shown. The Equation (14) is comparable with (11) if:

$$\left.\begin{array}{l}\psi(\ddot{x}) \text{ replaces } a_1 \\ g(x,\dot{x},\ddot{x},\dddot{x}) \text{ replaces } a_2 \\ \theta(x) \text{ replaces } a_3\dot{x} \\ f(x) \text{ replaces } a_4 x\end{array}\right\}$$

The function $\varphi(x)$ replacing a_4x suggests $x^{-1}\varphi(x)$ $(x \neq 0)$ replacing a_4, so that in turn (9) and (10) suggest that the existence of 2π periodic solution for (14) might be provable for arbitrary ψ *and* θ under the following hypotheses:

(H_1) $a_2 \leq 0$ and $\varphi(x)$ subject to the condition $x^{-1}\varphi(x) > 0$ or if $\varphi'(x)$ exists, for the stronger condition $\varphi'(x) > 0$

(H_2) $a_2 = 2N^2$ for some integer $N > 0$ with $\varphi(x)$ subject to the condition $x^{-1}\varphi(x) > N^4$ $(x \neq 0)$ or $\varphi'(x)$ exists, for the stronger condition $\varphi'(x) > N^4$

We now have the following theorem:

Theorem 1

Suppose ψ, h, θ, φ *and* p in Equation (14) are continuous and there are constants δ_0, δ_1, a_2 with

(i) $$\delta_0 = \begin{cases} o, & if \quad a_2 \leq 0 \\ \frac{1}{4}a_2^2, & if \quad a_2 > o \end{cases} \qquad \textbf{(15)}$$

and $\delta_1 > \delta_0$, such that,

(ii) $$a_2 > \inf \varphi'(x) \geq \delta_1 \qquad \textbf{(16)}$$

and $|\varphi(x)| \to +\infty \;\; as \;\; |x| \to \infty$

(iii) The function h is such that

$$|h(x,\dot{x},\ddot{x},\dddot{x})| \leq a_2 \qquad \textbf{(17)}$$

(iv) The function p is bounded and 2π periodic in t. That is:

$$p(t,x,\dot{x},\ddot{x},\dddot{x}) = p(t+2\pi,x,\dot{x},\ddot{x},\dddot{x}).$$

Then Equation (14) has at least one 2π periodic solution for arbitrary ψ and θ.

GENERAL COMMENTS ON SOME NOTATIONS

Throughout the proof, which follows, the capitals C, $C_1, C_2, \ldots$ represent positive constants whose magnitude depends at most on ψ, h, θ, φ *and* p. The $C's$ without suffixes are not necessarily the same in each place of occurrence but $C_1, C_2, \ldots$ with suffixes attached retain their identities throughout the proof of Theorem 1.

The symbol $|.|_\infty$, $|.|_1$, and $|.|_2$, in respect of the mapping: $[0 \quad 2\pi] \to \Re$ shall have their usual meanings.

Thus, for example, given a function

$$\theta : [0, \quad 2\pi] \to \Re$$

$$|\theta|_\infty = \max_{0 \leq t \leq 2\pi} |\theta(t)|, \qquad |\theta|_1 := \int_0^{2\pi} |\theta(t)| dt,$$

$$|\theta|_2 := \left(\int_0^{2\pi} \theta^2(t)\, dt\right)^{\frac{1}{2}} \geq 0$$

THE PROOF OF THEOREM 1

The proof of Theorem 1 is by the Leray-Schauder fixed point technique, Leray and Schauder (1934) and the starting point is a parameter $\lambda -$ dependent equation $(0 \leq \lambda \leq 1)$:

$$x^{(4)} + \lambda\psi(\ddot{x})\dddot{x} + h_\lambda(x,\dot{x},\ddot{x},\dddot{x})\ddot{x} + \lambda\theta(\dot{x}) + \varphi_\lambda(x) = \lambda p \quad \textbf{(18)}$$

where

$$h_\lambda(x,\dot{x},\ddot{x},\dddot{x})\ddot{x} = (1-\lambda)a_2\ddot{x} + \lambda h(x,\dot{x},\ddot{x},\dddot{x})\ddot{x}$$

and

$$\varphi_\lambda(x) = (1-\lambda)\delta_1 x + \lambda\varphi(x).$$

By setting

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = u,$$
$$\dot{u} = -\lambda\psi u - h_\lambda z - \lambda\theta(y) - \lambda\varphi(x) + \lambda p \quad \textbf{(19)}$$

Equation (19) can be rewritten compactly in the matrix form,

$$\dot{X} = AX + \lambda F(X,t) \quad \textbf{(20)}$$

where,

$$X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix},\quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\delta_1 & 0 & -a_2 & 0 \end{bmatrix},\quad F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Q \end{bmatrix} \quad \textbf{(21)}$$

with

$$Q = p(t) - \psi u - hz + a_2 z - \theta(y) - \varphi(x) + \delta_1 x.$$

We remark that Equation (18) reduces to a linear equation:

$$x^{(4)} + a_2\ddot{x} + \delta_1 x = 0 \quad \textbf{(22)}$$

when $\lambda = 0$ and to Equation (14) when $\lambda = 1$.

The eigenvalues of A defined by (21) are the roots of the auxiliary Equation (22) which is,

$$r^4 + a_2 r^2 + \delta_1 = 0 \quad \textbf{(23)}$$

The roots of the Equation (23) are not of the form $i\beta$ (β an integer) if

$$\delta_1 > \frac{1}{4}a_2^2 \quad \textbf{(24)}$$

Therefore the matrix $(e^{-2\pi A} - I)$ (I being the identity 4×4 matrix) is invertible.

Thus $X = X(t)$ is a 2π periodic solution of (21) if and only if X satisfies the equation:

$$X = \lambda TX,\quad \theta \le \lambda \le 1 \quad \textbf{(25)}$$

where the transformation T is defined by:

$$(TX)(t) = \lambda\int_t^{t+2\pi}(e^{-2\pi A} - I)^{-1} e^{A(t-s)} F(X(s),s)\,ds \quad \textbf{(26)}$$

[Jack Hale, 1963].

Let S be the space of all real continuous and 4-vector function $\bar{X}(t) = (x(t), y(t), z(t), u(t))$ which are of periodic 2π and with norm $\|\bar{X}\|_s$ defined by:

$$\|\bar{X}\|_s = \sup_{0\le t\le 2\pi}\{|x(t)| + |y(t)| + |z(t)| + |u(t)|\} \quad \textbf{(27)}$$

If the operator T is a compact mapping of S into itself, it suffices that for the proof of Theorem 1 to establish a priori bounds:

$$|x|_\infty,\ |\dot{x}|_\infty,\ |\ddot{x}|_\infty,\ |\dddot{x}|_\infty.$$

Next, we show the condition of Schaefer's Lemma (Schaefer 1955) are satisfied under the hypothesis of Theorem 1. This requires the proof of the following:

LEMMA 1.2

The transformation $T : S \to S$ is compact subject to the conditions of Theorem 1 if there exists an "a priori" bound,

$$\|X\|_s \le C_0,\quad \forall X \in S$$

satisfying (25) where C_0 is a fixed constant whose magnitude is independent of λ (Tejumola, 1966).

Finally, the proof of Theorem 1, will amount to concentrating on the Equation (18) and to prove that there exists constants C_1, C_2, C_3, C_4 independent of λ such that,

$$|x|_\infty \le C_1,\ |\dot{x}|_\infty \le C_2,\ |\ddot{x}|_\infty \le C_3,\ |\dddot{x}|_\infty \le C_4 \quad \textbf{(28)}$$

VERIFICATION OF EQUATION 28

Let $x(t)$ be a possible 2π periodic solution; then the main tool to be used here in the verification of Equation 18 is the function $V(x,y,z,u)$ defined by:

$$V = \lambda\int_0^z s\psi(s)\,ds + uz + y\varphi_\lambda + \lambda\int_0^y \theta(s)\,ds \quad \textbf{(29)}$$

The time derivative $\dot{V}$ of (29) along the solution paths of (19) is

$$\dot{V} = u^2 - hz^2 + y^2\varphi_\lambda' + hpz\,. \quad \textbf{(30)}$$

In dealing with term like $y^2\varphi_\lambda'(x)$ in which $\varphi_\lambda'(x)$ is positive only when $|x|$ is large, consider the function W defined by:

$$W = yH(x) \quad \textbf{(31)}$$

where,

$$H(x) = \begin{cases} \sin\left(\frac{\pi x}{4}\right), & |x|\le 2 \\ \operatorname{sgn} x, & |x| > 2 \end{cases}$$

Along the solution paths of (19),

$$\frac{d}{dx}\big(yH(x)\big) = y^2H'(x) + zH(x) \quad \textbf{(32)}$$

By considering the function:

$$\cup = V + \lambda C_0 yH(x) \quad \textbf{(33)}$$

and along the solution paths of (19), we shall have that:

$$\dot{U} = u^2 - hz^2 + y^2\varphi_\lambda' + \lambda pz + \lambda C_0 y^2 H'(x) + \lambda C_0 zH(x)$$

or,

$$\dot{U} = u^2 - hz^2 + y^2\left(\varphi_\lambda'(x) + \lambda C_0 H'(x)\right) + \lambda pz + \lambda C_0 zH(x) \quad \textbf{(34)}$$

Since $|H| \le 2\ \ \forall x$ and $H'(x) + \ge 0\ \ \forall x$ but $H'(x) \ge \dfrac{\pi}{4\sqrt{2}}$ when $|x| \le 1$, it follows from (34) that C_0 in (33) is fixed and large enough. We shall have that for every possible 2π periodic solution of (18) that:

$$\int_0^{2\pi} u^2 dt - \int_0^{2\pi} hz^2 dt + \int_0^{2\pi} y^2\left(\varphi_\lambda'(x) + \lambda C_0 H'(x)\right)dt = \int_0^{2\pi} -\lambda pz dt - \int_0^{2\pi} \lambda C_0 zH(x)\,dt \quad \textbf{(35)}$$

or

$$\int_0^{2\pi} u^2 dt + \int_0^{2\pi} |-h|\, z^2 dt + \int_0^{2\pi} y^2\left(\varphi_\lambda' + \lambda C_0 H'(x)\right)dt \le \int_0^{2\pi}\left(|-\lambda p| + |-\lambda C_0 H_x|\right)|z|\,dt$$

$$\int_0^{2\pi} u^2 dt + \int_0^{2\pi} a_2 z^2 dt + C_1\int_0^{2\pi} y^2 dt \le C_2\int_0^{2\pi} |z|\,dt \quad \textbf{(36)}$$

In particular,

$$\int_0^{2\pi} u^2 dt \le C_2 \int_0^{2\pi} |z| dt \qquad \textbf{(37)}$$

We have used here the fact that p is bounded and condition (17) in obtaining (36). Recall that $y = \dot{x},\ z = \ddot{x},\ u = \dddot{x}$, then Equation (37) implies

$$\int_0^{2\pi} \dddot{x}^2 dt \le C_2 \int_0^{2\pi} |\ddot{x}| dt$$
$$\le C_2 (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} \ddot{x}^2 dt \right)^{\frac{1}{2}} \qquad \textbf{(38)}$$

by Schwartz's inequality.

Consider the Fourier expansion of $x(t)$,

$$x'(t) \approx a_0 + \sum_{r=1}^{\infty} (a_r \cos rt + b_r \sin rt) \qquad \textbf{(39)}$$

From which we have

$$\ddot{x}(t) \approx -\sum_{r=1}^{\infty} r^2 (a_r \cos rt + b_r \sin rt) \qquad \textbf{(40)}$$

and

$$\dddot{x}(t) \approx \sum_{r=1}^{\infty} r^3 (a_r \sin rt - b_r \cos rt). \qquad \textbf{(41)}$$

Because of the standard Parseval's inequality, $\int_0^{2\pi} \ddot{x}^2 dt \le \int_0^{2\pi} \dddot{x}^2 dt$, which implies that:

$$\int_0^{2\pi} \dddot{x}^2 dt \le C_2 (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} \ddot{x}^2 dt \right)^{\frac{1}{2}} \le C_2 (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} \dddot{x}^2 dt \right)^{\frac{1}{2}}$$

by (38). Thus,

$$\left(\int_0^{2\pi} \dddot{x}^2 dt \right)^{\frac{1}{2}} \le C_2 (2\pi)^{\frac{1}{2}} \equiv C_3 \qquad \textbf{(42)}$$

Now $\dot{x}(0) = \dot{x}(2\pi)$ implies that there exists $\tau \in [0, 2\pi]$ such that $\ddot{x}(\tau) = 0$.

Thus, the identity $\ddot{x}(t) = \ddot{x}(\tau) + \int_\tau^t \dddot{x} ds$ holds.

That is, $\ddot{x}(t) = \int_\tau^t \dddot{x} ds$.

Therefore,

$$\max_{0 \le t \le 2\pi} |\ddot{x}(t)| \le \int_0^{2\pi} |\dddot{x}| dt \le (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} \dddot{x}^2 dt \right)^{\frac{1}{2}}$$

by Schwartz's inequality.

By (42), we have,

$$\max_{0 \le t \le 2\pi} |\ddot{x}(t)| \le C_3 (2\pi)^{\frac{1}{2}} \equiv C_4.$$

Thus,

$$|\ddot{x}|_\infty \le C_4 \qquad \textbf{(43)}$$

Again since $x(0) = x(2\pi)$ implies that there exists $\tau \in [0, 2\pi]$ such that $\dot{x}(\tau) = 0$ and the identity $\dot{x}(t) = \dot{x}(\tau) + \int_\tau^t \ddot{x} ds$.

Therefore,

$$\max_{0 \le t \le 2\pi} |\dot{x}(t)| \le \int_0^{2\pi} |\ddot{x}| ds \le (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} \ddot{x}^2 dt \right)^{\frac{1}{2}}$$

by Schwartz's inequality. From (43) $\max_{0 \le t \le 2\pi} |\dot{x}(t)| \le 2\pi C_4 \equiv C_5$.

That is,

$$|\dot{x}|_\infty \le C_5 \qquad \textbf{(44)}$$

From section 2, $|\theta|_2 = \left(\int_0^{2\pi} \theta^2(t) dt \right)^{\frac{1}{2}}$.

Therefore (42) implies that,

$$|\dddot{x}^2|_2 \le C_3 \qquad \textbf{(45)}$$

But,

$$\max_{0 \le t \le 2\pi} |\dddot{x}(t)| \le \int_0^{2\pi} |\dddot{x}| dt \le (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} \dddot{x}^2 dt \right)^{\frac{1}{2}}$$

by Schwartz's inequality.

$\max_{0\le t\le 2\pi}\left|\ddot{x}(t)\right| \le (2\pi)^{\frac{1}{2}} C_3 \equiv C_6$ by (45). Thus,

$$\left|\ddot{x}\right|_\infty \le C_6 \qquad \textbf{(46)}$$

There remains only the first inequality for (28) to be fully established. Integrate Equation (18) with respect to t from $t = 0$ to $t = 2\pi$.

$$\int_0^{2\pi} x^{(4)} dt + \int_0^{2\pi} \lambda\psi(\ddot{x})\dddot{x}\, dt + \int_0^{2\pi} h_\lambda(x,\dot{x},\ddot{x},\dddot{x})\,\ddot{x}\, dt + \int_0^{2\pi} \lambda\theta(\dot{x})\, dt + \int_0^{2\pi} \varphi_\lambda(x)\, dt = \int_0^{2\pi} \lambda p\, dt$$

using the 2π periodicity condition on $x(t)$, we have the following:

$$\int_0^{2\pi} \lambda\psi(\ddot{x})\dddot{x}\, dt + \int_0^{2\pi} h_\lambda(x,\dot{x},\ddot{x},\dddot{x})\,\ddot{x}\, dt + \int_0^{2\pi} \lambda\theta(\dot{x})\, dt + \int_0^{2\pi} \varphi_\lambda(x)\, dt = \int_0^{2\pi} \lambda p\, dt$$

or

$$\int_0^{2\pi} \varphi_\lambda(x)\, dt = \int_0^{2\pi} \lambda p\, dt - \int_0^{2\pi} \lambda\psi(\ddot{x})\dddot{x}\, dt - \int_0^{2\pi} h_\lambda(x,\dot{x},\ddot{x},\dddot{x})\,\ddot{x}\, dt - \int_0^{2\pi} \lambda\theta(\dot{x})\, dt \qquad \textbf{(47)}$$

By Equations (43), (44) and (46), and the continuity of p and 2π periodic in t, the right hand side of Equation (47) is bounded. That is,

$$\left|\int_0^{2\pi} \lambda p\, dt - \int_0^{2\pi} \lambda\psi(\ddot{x})\dddot{x}\, dt - \int_0^{2\pi} h_\lambda(x,\dot{x},\ddot{x},\dddot{x})\,\ddot{x}\, dt - \int_0^{2\pi} \lambda\theta(\dot{x})\, dt\right|$$

$$\le \left|\int_0^{2\pi} \lambda p\, dt\right| + \left|\int_0^{2\pi} \lambda\psi(\ddot{x})\dddot{x}\, dt\right| + \left|\int_0^{2\pi} h_\lambda(x,\dot{x},\ddot{x},\dddot{x})\,\ddot{x}\, dt\right| + \left|\int_0^{2\pi} \lambda\theta(\dot{x})\, dt\right| \le C_7$$

That is,

$$\left|\int_0^{2\pi} (1-\lambda)\delta_1 x dt + \int_0^{2\pi} \lambda\varphi(x) dt\right| \le C_7 \qquad \textbf{(48)}$$

But (*) implies

Given $\alpha > 0$ $\exists$ $\beta > 0$ such that

$$|x| > \beta \Rightarrow |h(x)| > \alpha$$

Then $\exists$ a $\tau \in [0, 2\pi]$ such that

$$|x(\tau)| \le C_8 \qquad \textbf{(49)}$$

Now (i) if $x(\tau) = 0$ then we are done.

(ii) suppose NOT i.e. $x(\tau) \ne 0$ for any τ , then left hand side of (48)

$$\int_0^{2\pi}(1-\lambda)\delta_1|x|dt+\int_0^{2\pi}\lambda|\varphi(x)|dt$$
$$>\int_0^{2\pi}(1-\lambda)\delta_1\beta dt+\int_0^{2\pi}\lambda\alpha dt$$
$$>2\pi(1-\lambda)\delta_1\beta+2\pi\lambda\alpha$$

That is the left had side of (48) is not bounded. This is a negation to the boundedness in (48). Thus Equation (49) holds.

The Equation (49) implies that there exists $\tau\in[0,\ 2\pi]$ such that,

$$|x(\tau)|\le C_9 \qquad \textbf{(50)}$$

and the identity $x(t)=x(\tau)+\int_\tau^t\dot{x}\,ds$ holds. Therefore,

$$\max_{0\le t\le 2\pi}|x(t)|\le|x(\tau)|+\int_0^{2\pi}|\dot{x}|dt$$

$$\le C_9+(2\pi)^{\frac{1}{2}}\left(\int_0^{2\pi}\dot{x}^2dt\right)^{\frac{1}{2}}=C_9+(2\pi)^{\frac{1}{2}}|\dot{x}|_2$$

by Schwartz's inequality. From (44),

$$\max_{0\le t\le 2\pi}|x(t)|\le C_9+C_{10}\equiv C_{11}.$$

That is, $|x|_\infty\le C_{11}$.

The estimates (43), (44), (46) and (51) verify (28) and hence the proof of Theorem 1 follows.

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