Stability and Harmonic Oscillations in a Homogenous Third Order Differential Equation with Numerical Illustrations

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ABSTRACT

Some qualitative properties of a homogeneous third order differential equation ($\dddot{x}+a\ddot{x}+b\dot{x}+cx=0$) have been extensively used in determining stability, boundedness, and the existence of periodic solutions for the non-homogeneous third order equation. This idea has been extended to the non-linear differential equations of the third order. Little did we know that the homogeneous equation is in itself stable and harmonic subject to the conditions $a>0,\ b>0,\ ab>c$ or $a>0,\ b>0,\ ab<c$. This has been vividly shown by the use of the MATHCAD® software package.

(Keywords: constant coefficient equation, stability, harmonic oscillation, homogeneous equations, qualitative properties)

INTRODUCTION

Consider the constant coefficient homogeneous equation of third order differential equation

$$\dddot{x}+a\ddot{x}+b\dot{x}+cx=0 \quad \textbf{(1)}$$

where a, b, and c are constants. Earlier results in Ezeilo (1960), Pliss (1961), Villari (1964), Ezeilo (1975), have depended on qualitative properties of a, b, and c for existence of periodic solutions for the non-homogeneous third order differential, where the forcing term may be periodic or not, but not zero. Some of these properties were the Routh Hurwitz's conditions,

$$a>0,\ \ b>0,\ \ ab>c \quad \textbf{(2)}$$

or the non-Routh Hurwitz's conditions,

$$a>0,\ \ b>0,\ \ ab<c \quad (3)$$

[See Tejumola (1966), Ezeilo (1975)]. So far the conditions (2) were necessary and sufficient conditions for the global asymptotic stability of the zero solutions of Equation (1). Little was known about harmonic oscillations of equation (1) subject to (2) or (3).

In these present circumstances our objective is to propose a theorem that would guarantee stability and harmonic oscillations for Equation (1) subject to Equation (2) or (3). Thus we have the following:

THEOREM 1:

Suppose that in Equation (1) the conditions (2) or (3) hold. Then the solution to Equation (1) is stable and harmonic.

REMARK: The proof of Theorem 1 is equivalent to showing that Equation (1) is stable and also harmonic subject to the conditions (2) or (3), and we proceed as follows:

PROOF OF STABILITY OF EQUATION (1)

The proof of stability of solutions in Equation (1) subject to the condition (2) is by the use of the Lyapunov function and its appropriate Lyapunov theorem. Thus consider Equation (1) or the equivalent system:

$$\dot{x}=y,\ \ \dot{y}=z,\ \ \dot{z}=-az-by-cx \quad \textbf{(4)}$$

where a, b, and c are subject to condition (2). Now we need to construct a suitable Lyapunov function for the system (4). Take v to be of the form:

$$2V = k_1x^2 + k_2y^2 + k_3z^2 + 2k_4xy + 2k_5yz + 2k_6xz \qquad \textbf{(5)}$$

where $k_i \quad i = 1, 2, ..., 6$ are constants yet to be determined. The time derivative along the solution paths of (4) is:

$$\begin{aligned}\dot{V} &= (k_1 - k_6b - k_5c)xy + (k_2 + k_6 - k_3b - k_5a)yz \\ &+ (k_4 - k_3c - k_6a)xz - k_6cx^2 \\ &- (k_5b - k_4)y^2 - (k_3a - k_5)z^2 \qquad \textbf{(6)}\end{aligned}$$

We distinguish the following cases for (6).

(I) $\dot{V} = -\alpha x^2,\ \dot{V} = -\alpha y^2,\ \dot{V} = -\alpha z^2$

(II) $\dot{V} = -\alpha(x^2 + y^2),\ \dot{V} = -\alpha(y^2 + z^2),\ \dot{V} = -\alpha(x^2 + z^2)$

(III) $\dot{V} = -\alpha(x^2 + y^2 + z^2)$.

Here we consider the case where $\dot{V} = -\alpha y^2$ and choose the coefficients $xy,\ xz,\ x^2,\ z^2$ in (6) to be zero. That is:

$$\left.\begin{aligned} &k_1 - k_6b - k_5c = 0 \ ....(i) \\ &k_2 + k_6 - k_3b - k_5a = 0..........(ii) \\ &k_4 - k_3c - k_6a = 0................(iii) \\ &k_3a - k_5 = 0.........................(iv) \\ &k_5b - k_4 > 0..........................(v) \\ &k_6 = 0.................................(vi) \end{aligned}\right\} \qquad \textbf{(7)}$$

From (7): (iv) implies that $k_5 = ak_3$

(i) implies that $k_1 = ack_3$

and subsequently $k_2 = bk_3 + a^2k_3$

$k_4 = ck_3$

$abk_3 - ck_3 > 0$

implies that $(ab - c)k_3 > 0$. That is $ab - c > 0$ which that $ab > c$. Setting $k_3 \equiv 1$ and substituting for $k_{i's}$ in terms of k_3, we have:

$$2V = acx^2 + (b + a^2)y^2 + z^2 + 2cxy + 2ayz$$

$$= ac(x + a^{-1}y)^2 + (b - a^2 - ca^{-1})y^2 + (z + ay)^2 \qquad \textbf{(8)}$$

Which is positive definite since $ab > c > 0.$ But

$$\dot{V} = -(ab - c)y^2 \qquad \textbf{(9)}$$

Clearly (9) is negative definite since $ab - c > 0$. We state hereunder a theorem due to A.M. Lyapunov.

THEOREM:

Suppose that there exist a C^1 function $V : \mathbf{R}^3 \to \mathbf{R}$ such that:

(i) V is positive definite

(ii) $\dot{V}$ is negative definite

Then the trivial solution is asymptotically stable in the sense of Lyapunov.

Clearly our V defined by (8) is positive definite and the time derivative $\dot{V}$ is negative definite. Therefore the hypotheses of Lyapunov theorem are satisfied and stability of Equation (1) is established. Next we establish stability and harmonic oscillations for Equation (1) using the conditions (2) or (3). This part of the proof which follows in the next section is done by illustration using the MATHCAD® software.

MATHCAD®

MATHCAD® software is a life document interface of spreadsheet with "WYSIYG" meaning "what you see is what you get" interface of a word processor. It allows us to solve a wide range of technical problems from simple ones to complicated cases. We can place equations, texts and graphs in the MATHCAD® work sheet where the calculations are done and the results represented in any dimensions. The MATHCAD® solves the mathematical problems the way we do them analytically. Instead of using programming like syntax, MATHCAD® uses programming language equations like this:

$$X = \left(-B + SQRT\left(B^{**2} - 4^{*}A^{*}C\right)\right) {}^{1}\!/_{2}{}^{*} A \quad \textbf{(10)}$$

in the spread sheet, this will go into cells in the form:

$$X = +\left(-B_1 + SQRT\left(B^{*}{}_{1}B_1 - 4^{*}A_1{}^{*}c_1\right)\right) {}^{1}\!/_{2}{}^{*} A_1 \quad \textbf{(11)}$$

However MATHCAD® equations are the same way they are in the text, e.g.

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \textbf{(12)}$$

The only difference is that MATHCAD® equations and graphs are alike. That is if there will be any change in the data variables the MATHCAD® software recalculates and redraws the graphs instantly. We can thus visualize the calculations and graphs. It is this unique feature of the MATHCAD® that we will utilize to illustrate stability and harmonic oscillations for equation (1) subject to conditions (2) or (3).

DISCUSSIONS

The graphical solutions of the differential equations:

$$\dddot{x} + 2\ddot{x} + 2\dot{x} + 3x = 0 \quad \textbf{(13)}$$

and

$$\dddot{x} + 2\ddot{x} + \dot{x} + 3x = 0 \quad \textbf{(14)}$$

Which were given in Figure I and Figure II are symmetrical. The graphical solutions to Equation (13) is harmonic from the starting point and stable at the right hand side of the graph while that of Equation (14) is stable from the beginning and harmonic at the right hand side of the graph. That is the graph of (13) and (14) interchange positions for stability and harmonic oscillations. The changes are due to difference in equations (1) and (3). Therefore Equation (1) is stable and harmonic subject to condition (2) or (3).

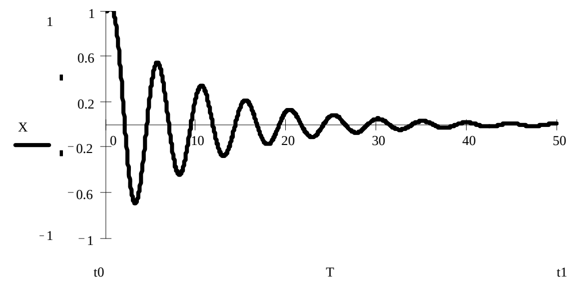

Figure 1: Graphical Solution for the Equation $\dddot{x} + 2\ddot{x} + 2\dot{x} + 3x = 0$

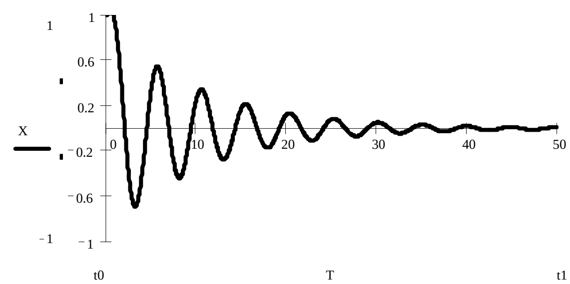

Figure 2: Graphical Solution for the Equation $\dddot{x} + 2\ddot{x} + \dot{x} + 3x = 0$

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SUGGESTED CITATION:

Ogbu, H.M. 2007. "Stability and Harmonic Oscillations in a Homogenous Third Order Differential Equation with Numerical Illustrations". *Pacific Journal of Science and Technology*. 8(1):126-129.

Pacific Journal of Science and Technology