# Assessment of the use of Gaussian Beam Approach in Studying Wave Propagation through Materials

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## ABSTRACT

In this work, the authors examine the electromagnetic wave propagation through a medium with the concept of Gaussian Beam. The general wave equation was used with a solution carried out in terms of the electric field and intensity distributions approximated to Gaussian Function, and it is with this, we analyze the dependency of distance of propagation on Gaussian beam distribution spread considering the distant from the axis at which the intensity of the beam distribution begins to fall at a given estimate, in relation to its peak value. The influence of the optimum beam waist, and the beam spread with respect to the intensity distribution, also was taken into consideration by the distribution profiles as depicted by the plotted profiles using different waist with an indication that the Gaussian profiles changes with beam waist.

(Keywords: electromagnetic wave, medium, Gaussian beam, distribution Gaussian function, intensity, beam waist)

## INTRODUCTION

The study of beam propagation has been key research in the past and at present time [1-4] especially in this current era with evolution of photonics along with the rapid evolution and application of fiber optics, integrated optics, and applications of lasers in both the medical sciences and technology. Because of these factors, there has been a growing interest in the study of Gaussian beam propagation [5]. This is based on the fact that Gaussian beam relates to focusing and modification of the shape of the propagating electromagnetic wave or laser beam.

From earlier findings on the research on laser beams, it has been found that laser beam propagation can be approximated by an ideal Gaussian beam intensity profile. Understanding of the basic properties of Gaussian beams has been specifically found to be vital for selecting the best optics for practical application.

Sequel to this, many scientists had worked on Gaussian beam applications in electromagnetic wave propagation and in optics. For instance, a work has been carried out on non-specular phenomena for beam reflection at monolayer and multilayered dielectric interface respectively from where it has revealed that under various conditions non-specular beam phenomena is more realizable and on his own presented a unified and simplified analysis of the lateral and longitudinal displacement with angular deflection on reflection of a Gaussian beam at a dielectric interface with two or more layer, [6-11].

However, in this paper, we intend to study electromagnetic wave propagation using the concept of a Gaussian beam starting from general wave equation, with which is obtained in conjunction with the Gaussian function in which waves that operate on the fundamental transverse mode is approximated to Gaussian profile with distribution profile assessed with intent to observe the influence of the beam waist on the profile.

## THEORETICAL FRAMEWORK

In this case we start with the wave equation in terms of electric field given as:

$$\nabla^2 E - \mu\varepsilon_r\varepsilon_0\omega^2 \frac{\partial^2 E}{\partial t^2} = 0 \qquad \textbf{(1)}$$

where $\varepsilon_r$ is the relative dielectric constant as a function wave solution of the form:

$$\psi(r) = \psi \exp i(\omega t - k_r(r) \qquad \textbf{(2)}$$

Considering a medium with variation in refractive index, the solution of the above equation gives:

$$\nabla^2\psi + \mu\varepsilon_o\varepsilon_r\omega = 0 \qquad \textbf{(3)}$$

Where k signifies the propagation constant in the medium as given by:

$$k = \omega\varepsilon\ \mu\varepsilon_o \qquad \textbf{(4)}$$

With Equation (4) and (2), the result yields:

$$\nabla^2\psi + k^2(r)\psi = 0 \qquad \textbf{(5)}$$

If the medium is an absorbing or the type exhibits gain, then its relative dielectric constant, $\varepsilon_r$ is considered to have real and imaginary part, but in a situation where the medium conducts with conductivity given by $\delta$. then the complex propagation vector is invariably introduced which is required to obey the rule that:

$$k^2 = \omega^2\mu\varepsilon_r\left(I - i\frac{\delta}{\omega\varepsilon_r\varepsilon_o}\right) \qquad \textbf{(6)}$$

[Ralston, 1982].

However, a simple solution of this type is inadequate to describe the field distributions with transverse mode and as a result, and as a result, a solution that will satisfy a plane wave equation of the form is sought:

$$\psi = \psi_o(x, y, z)\exp ik_z \qquad \textbf{(7)}$$

Specifying field propagating in the z-direction and then localized in the z-axis. Thus, the idea is to obtain a solution of the wave equation that gives phase front that can be approximated over a narrow region within the z-direction. Thus, substituting Equation [7] into Equation (4) we obtain:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - 2ik\frac{\partial^2\psi}{\partial z^2} = 0 \qquad \textbf{(8)}$$

Equation 8 signifies a paraxial beam-like solution where $\psi$ varies slowly with z-direction and with this equation becomes:

$$\psi_o(x, y) = \exp[-ip(z) + \frac{k}{q(z)}r^2] \qquad \textbf{(9)}$$

Where $r^2 = x^2 + y^2$ is the square distant of the points $x, y$ from the axis of propagation where $p(z)$ and $q(z)$ are beam parameters.

Equation (9) gives the fundamental two-dimensional Gaussian beam of time-independent wave equation which explains the behavior of the field propagating through z-direction at every position along the z-axis.

## Intensity of Gaussian Beam

The intensity, $I_{(x,y)}$ of the Gaussian beam at any given point is as in equation given below:

$$I(x, y) = \psi\psi^* = \exp\left[-ip(z) + \frac{k}{2q(z)}r^2\right]\exp\left(-ip^*_{(z)} + \frac{k}{2q^*_{(z)}}\right) \qquad \textbf{(10)}$$

Where $p^*_{(z)}$ and $q^*_{(z)}$ are the complex conjugate of $p_{(z)}$ and $q_{(z)}$ respectively and hence:

$$\psi\psi^* = \exp\left(-ip(z) - p^*(z)\right)\exp-\frac{ikr^2}{z}\left(\frac{1}{q(z)} + \frac{1}{q^*(z)}\right) \qquad \textbf{(11)}$$

Considering the two parameters of the real beam as $R_{(z)}$ and $\omega$ relating being related to $q_{(z)}$ and dependent on z:

$$\frac{1}{q(z)} = \frac{1}{R} - \frac{i\lambda}{\pi\omega^2} \qquad \textbf{(12)}$$

And $p(z)$ being the part that expresses the axial distance from the waist of Gaussian beam, the product of $\psi\psi^*$ can be written as:

$$\frac{ikr^2}{2}\left(\frac{-i\lambda}{\pi\omega^2}-\frac{i\lambda}{\pi\omega^2}\right) \quad \textbf{(13)}$$

$$I(x,y)\alpha\exp\left(\frac{-2r^2}{\pi\omega^2}\right) \quad \textbf{(14)}$$

Signifying the dependency of Gaussian beam intensity on r while omega signifies the distance from the axis at z when the intensity of the Gaussian beam falls as $1/e$ of its peak value on the axis $p_{(z)}$ and the total power of the beam with these parameters as in Equation 10, becomes:

$$\psi=\exp\left[-i(kz+p(z))+\frac{kr^2}{2}\left(\frac{1}{R}-\frac{i\lambda}{\pi\omega^2}\right)\right] \quad \textbf{(15)}$$

$p_{(z)}$ is evaluated to give

$q \neq q_0 + z$,

With $q_0$ as a constant of integration that expresses the value of the beam parameter at the plane $z=0$ for which when we differentiate:

$$\frac{dp}{dz}=\frac{i}{q_o+qz},$$

we obtain:

$$p_{(z)}=i\left[\text{In } z+q_0\right]-\left(\theta+i\text{ In } q_0\right)\right] \quad \textbf{(17)}$$

Where $\theta - i\text{ In } q_0$ defines a constant of integration and then when we substitute Equation (17) into (15) we have:

$$\psi=\exp\left\{-\left[kz-iIn\left(1+\frac{z}{q_o}\right)+\frac{kr^2}{z}\left(\frac{1}{R}-\frac{i\lambda}{\pi\omega^2}\right)\right]\right\} \quad \textbf{(18)}$$

The factor $\exp p^{-i\theta}$ is a constant phase factor.

**<u>The Effect of Diffraction on Gaussian</u>**

The limitations of a Gaussian beam is that even if the wave fronts was made flat at some plane, it quickly turns out in form of curvature and begin to spread in accordance with:

$$R_{(z)}=z\left[1+\left(\frac{\pi w_0^2}{\lambda z}\right)^2\right] \quad \textbf{(19)}$$

and

$$w_{(z)}=w_0\left[1+\left(\frac{\lambda z}{\pi w_0^2}\right)^2\right]^{\frac{1}{2}} \quad \textbf{(20)}$$

where $z$ is the distance propagated from the plane where the wave-front is flat $\lambda =$ wavelength of light, $w$ is the radius of the $\frac{1}{e^2}$ irradiance contour at the plane where wave-front is flat, $w_{(z)}$ is the radius of the $\frac{1}{e^2}$ contour after the wave has propagated a distance $z$.

Considering the optimum starting beam radius for a distance $z$, we have $\omega_o=\left(\frac{\lambda z}{\pi}\right)$, Equation 20 can be reduced to:

$$\omega(z)=3/2\,\omega_o \quad \textbf{(21)}$$

With the irradiance distance distribution of the Gaussian beam described as:

$$I(y)=I_o\exp-\frac{-2r^2}{\omega^2}=\frac{2p}{\pi\omega^2}\exp-\frac{2r^2}{\omega^2} \quad \textbf{(22)}$$

Where $\omega=\omega(z)$ and $p$ is the total power in the beam which is the same at all cross sections of the beams which relates to:

$$In\frac{I(z)}{I_o}=\frac{-2r^2}{\omega^2}=-2r^2\frac{3\omega_o}{2} \quad \textbf{(23)}$$

The above Equation (23) can be reduced to:

$$In\frac{I_p}{I_o}=3\omega_o\left(x^2+y^2\right) \quad \textbf{(24)}$$

And at the optimum starting beam radius for a given distance z.

$$I_o=\frac{2p}{\pi\omega^2}\omega_0^2 \quad \textbf{(25)}$$

**DISCUSSION**

Like plane waves, Gaussian modes are characterized by normal modes of wave propagation in an isotropic, homogeneous medium though different from a plane wave, however because it has a finite cross-sectional field distribution defined by its spot size, and due to the fact that the divergence is inversely proportional to the spot size, for a given wavelength, a Gaussian beam that is focused to a small spot diverges rapidly as it propagates away from the focus. Therefore, in order to minimize the divergence of a Gaussian beam in the far field (and increase its peak intensity at large distances), it must have a large cross-section (w0) at the waist, and thus a large diameter where it is launched, since it is never less than w0. As observed in Figures 3-4, it could be seen that as the waist increased, the distribution profile changed patterns.

Therefore in order to appreciate the principles and limitations of Gaussian beam, it is first necessary to understand the nature of the electromagnetic wave characterized by a perfect coherent plane wave considered to have a Gaussian transverse irradiance characteristics as can be obtained in laser beam in which the Gaussian shape is truncated at some diameter either by internal dimensions of laser as in Figure 3 or by some limiting aperture in the optical train. This is specified by two common acceptable defined concepts: diameter at which the beam irradiance (intensity) has falls to or the axial value and the diameter at which the beam irradiance (intensity) axial value which is regarded as the full width at half maximum has fallen as shown in Figures 1-2.

In general, since diffraction causes transverse spreading of light waves as it propagates, it is therefore difficult to have perfectly collimated beam even in laser. Thus, the expression in Equation 25 describe accurately beam spreading, making it easy to generalize that even if the wave-front were made perfectly flat at some plane, it would quickly acquire curvature and begins spreading which depicted beam distribution in terms of percentage irradiance as a function of contour radius, and this why this relationship between beam width and divergence is a fundamental characteristic of diffraction and reflection [12-13], and of the Fourier transform which describes Fraunhofer diffraction is needed to be understood clearly as a beam with any specified amplitude profile also obeys this unique inverse relationship stressed here, but the fundamental Gaussian mode is a special case where the product of beam size at the focus and far-field divergence is smaller than for any other case.

In most cases, the Gaussian intensity profile for far-field divergence must be measured at a distance greater than the half width of intensity distribution while the near or mid-field divergence values are obtained by measuring a distance less than the half width of the intensity distribution as depicted in Figure 5. Therefore, in order to achieve optimum beam spread, and collimation over a distance, the optimum starting beam radius/waist must be determined for a particular irradiance. Though this depends strictly on the type of beam, the general expression for optimum starting beam radius for any given distance, z is $\omega_o(optimum)=\left(\frac{\lambda z}{\pi}\right)^{1/2}$ coupled with the axial distance within the Rayleigh range which given as given in Equation (22) with:

$$z_R=\frac{p\omega_o}{I}$$

The use of these values provide the best: combination for minimum starting beam diameter and minimum beam spread which is ratio of $\omega(z)$ to $\omega_o\left(\frac{\omega(z)}{\omega_o}\right)$ over a distance, z and a defined wave length It is also important to mention here that as $z >> z_R$, the width approaches a straight line and the central line of the axis of the beam may give a beam divergence which is expressed as $q >> \frac{1}{p_o \omega_0}$ from where the expression for the angular spread of the beam far from the waist can be written as $Q = 2q$. This of course depends on the type of radiation and the wavelength as earlier mentioned. However, from the foregoing analysis it is important to note that if the starting beam radius that can yield the minimum beam waist as shown in Figure 5 is used for a particular irradiance, the beam diameter and beam spread over a given over a distance, beam expanding optics hart allows such adjustment of the position of the beam waist is required for this to be accomplished [14-17].

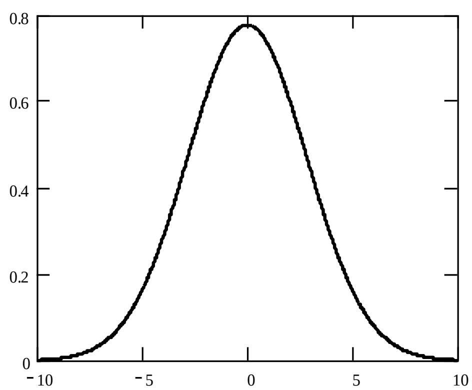

**Figure 1:** Gaussian Distribution Profile when y=2, $\omega = 3$.

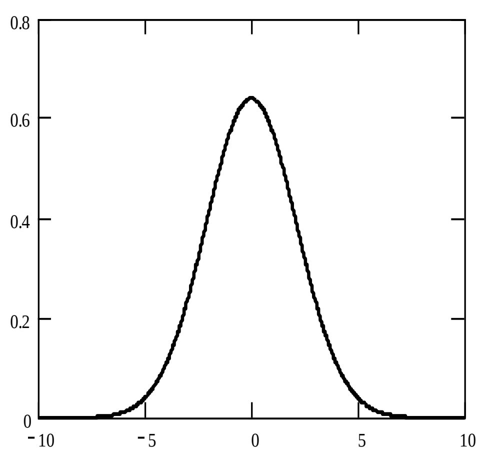

**Figure 2:** Gaussian Distribution Profile when y=2, $\omega = 4$.

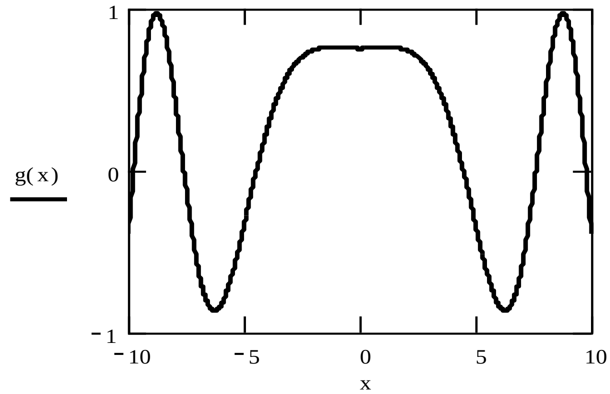


**Figure 3:** Gaussian Distribution Profile when y=2, $\omega = 5$.

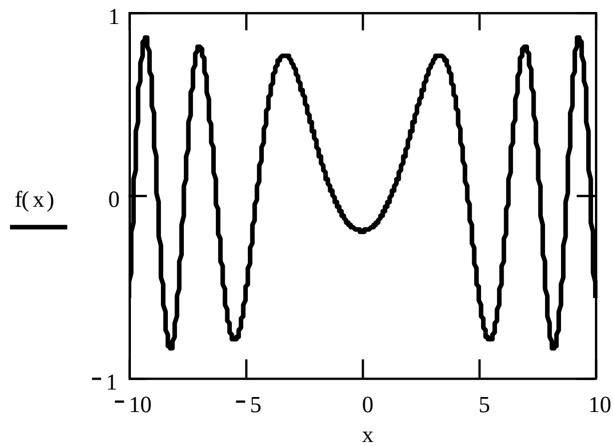


**Figure 4:** Gaussian Distribution Profile when y=2, $\omega = 10$.

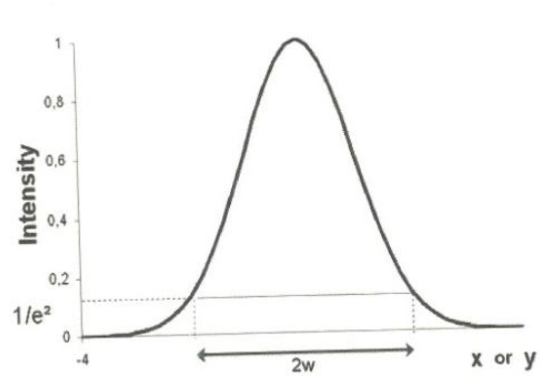


**Figure 6:** Gaussian Distribution Intensity as Function of width, $\omega$.

## CONCLUSION

Gaussian intensity profiles as observed from this study seem to manifest one feature that is based on the fact that beam waist, w plays a role in determining the distribution profile. This explains the fact that the far-field divergence must be measured at a distance greater than the half width of intensity distribution while the near or mid-field divergence values are obtained by measuring a distance less than the half width of the intensity distribution in order to obtain a good normal distribution profile.

Therefore, in order to achieve optimum beam spread, and collimation over a distance, the optimum starting beam radium/waist must be determined. Though one should note that this depends strictly on the type of beam that is whether the beam is coherent or not, the general expression for optimum starting beam radius for any given distance, z is:

$$\omega_o(optimum) = \left(\frac{\lambda z}{\pi}\right)^{1/2}.$$

Although the tails of a Gaussian function never actually reach zero, for the purposes of the following discussion the "edge" of a beam is considered to be the radius where $r = \omega(z)$. That is where the intensity has dropped to $1/e^2$ of its on-axis value.

Now, for $z >> z_R$, the parameter $\omega(z)$ increases linearly with $z$ which implied that far from the waist, the beam "edge" in the above sense tend to form a cone-shaped figure. The angle between that cone whose $r = \omega(z)$ and the beam axis $r = 0$ defines the divergence characteristics of the beam.

The study here explains the fact that the concept of Gaussian beam is realizable if the beam is focused on a small spot as it spreads out rapidly as it propagate away from the spot, and at a certain value of the beam waist, the distribution profile changes which implies that in order for the beam to be well collimated, it must have a specific value of diameter which is in relation to the diffraction and reflection characteristics of the beam [10-13]. Thus, in order to achieve normal Gaussian beam distribution, the product of the width and divergence of the beam profile must be as small as possible to be achieved when paraxial approximation is considered [14-16]. Thus, this implies that the idea of Gaussian beam model is realizable and valid only for beam whose width is larger than that of Equation 21.

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