Application of Least Squares Method for Solving Volterra Fractional Integro-Differential Equations Based on Constructed Orthogonal Polynomials

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ABSTRACT

The fractional derivative is considered in this paper as a numerical method for solving fractional integro-differential equations in Caputo sense. The Standard Least Squares Method (SLSM) with orthogonal polynomials generated as fundamental functions is the method proposed. This type of problem is reduced to the solution of a system of linear algebraic equations, which is then solved using MAPLE 18. We used numerical examples to demonstrate the accuracy and applicability of the proposed method. The method is simple to implement and accurate when used to solve fractional integration differential equations, according to numerical data.

(Keywords: standard least squares method, SLSM, orthogonal polynomials, linear algebra, fractional integration, fractional calculus, differential equations, FIDE, MAPLE 18)

INTRODUCTION

Fractional calculus is one of the most reliable tools for approaching complex systems with memory. Several definitions of fractional integrals and derivatives have been proposed in recent decades.

Integro–differential equations with fractional order derivatives appear in many mathematical formulations of real-world phenomena (Caputo, 1967), (Momani and Qaralleh 2006). As a result, these equations have attracted the attention of mathematicians and other scientists, and many researchers have studied the solutions of fractional integro–differential equations in recent years.

Many fractional integra-differential equations (FIDEs) are difficult to solve and may not have analytical or exact solutions in the interval of consideration, necessitating the use of approximate and numerical methods. Several numerical methods have been proposed to solve the FIDEs, including the Adomian Decomposition Method (Mittal and Nigam, 2008), the Standard Least Squares Method (Mohammed, 2014; Oyedepo, et al., 2016; Oyedepo and Taiwo, 2019), the homotopy analysis transform method (Mohamed, et al., 2016), and the collocation method (Rawashdeh, 2006).

The main objective of this work is to find the numerical solution of the Volttera type fractional integra differential equation using the standard less square method based on the orthogonals constructed as basic functions. The general form of the problem class considered in this work is as follow:

$$D^{\alpha}u(x) = p(x)u(x) + f(x) + \int_0^x k(x,t)u(x)dt, \quad o \leq x,t \leq 1, \tag{1}$$

With the following supplementary conditions:

$$u^{(i)}(0) = \delta_{i,} i = 0,1,2,\dots,n-1, \text{n} - 1 < \alpha \leq n, \; n \in, N \tag{2}$$

Where $D^{\alpha}u(x)$ indicates the $\propto th$ Caputo fractional derivative of $u(x)$; $p(x)$,$f(x)$, $K(x,t)$ are given smooth functions, δ_i are real constant, x and t are real variables varying [0, 1] and $u(x)$ is the unknown function to be determined.

Definition of Basic Terms

Definition 1.

Fraction Calculus involves differentiation and integration of arbitrary order (all real numbers and complex values).

Example $D^{\frac{1}{2}}, D^{\pi}, D^{2+i}$ etc.

Definition 2.

Gamma function is defined as:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \qquad (3)$$

This integral converges when real part of z is positive $(Re(z) \leq 0)$.

$$\Gamma(1+z) = z\Gamma(z) \qquad (4)$$

Where z is a positive integer:

$$\Gamma(z) = (z-1)! \qquad (5)$$

Definition 3.

Beta function is defined as:

$$B(v,m) = \int_0^1 (1-u)^{v-1} u^{m-1} du = \frac{\Gamma(v)\Gamma(m)}{\Gamma(v+m)} = B(v,m),$$
$$v, m \in R_+ \qquad (6)$$

Definition 4.

In this work, we defined absolute error as:

Absolute Error $= |U(x) - u_m(x)|; \quad 0 \leq x \leq 1, \qquad (7)$

where $U(x)$ is the exact solution and $u_m(x)$ is the approximate solution.

Where $u_m(x)$ Bernstein polynomial of degree m where

$$a_j, \mathrm{j} = 0,1,2,\ldots \qquad (8)$$

are constants.

Definition 5.

Riemann-Liouville fractional derivative defined as:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^m(s) ds, \qquad (9)$$

m is positive integer with the property that $m - 1 < \alpha < m.$

Definition 6.

The Caputo Factional Derivative is defined as:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^m(s) ds \qquad (10)$$

Where m is a positive integer with the property that $m - 1 < \alpha < m$

For example, if $0 < \alpha < 1$ the Caputo fractional derivative is:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} f^1(s) ds \qquad (11)$$

Hence, we have the following properties:

(1) $J^{\alpha} J^{v} f = j^{\alpha+v} f, \alpha, v > 0, f \in C_{\mu}, \mu > 0$

(2) $J^{\alpha} x^{\gamma} = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma,}$
$\alpha > 0, \gamma > -1, x > 0$

(3) J^{α}
$D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{k}(0)\frac{x^k}{k!}, \qquad x > 0, m-1 < \alpha \leq m$

(4) D^{α}
$J^{\alpha} f(x) = f(x), \quad x > 0, m-1 < \alpha \leq m,$

(5) $D^{\alpha}C = 0, C$ is the constant,

(6) $\begin{cases} 0, & \beta \in N_0, \beta < [\alpha], \\ D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, & \beta \in N_0, \beta \geq [\alpha], \end{cases}$

Where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0, 1, 2, \dots\}$

Definition 7.

Orthogonality: Two functions say $u_p(x)$ and $u_q(x)$ defined on the interval $a \leq x \leq b$ are said to be orthogonal if:

$$< u_p(x), u_q(x) = \int_a^b u_p(x)\, u_q(x)\, dx = 0 \qquad (12)$$

If, on the other hand, a third function $w(x) > 0$ exists such that:

$$< u_p(x), u_q(x) >= \int_a^b w(x) u_p(x)\, u_q(x)\, dx = 0 \qquad (13)$$

Then, we say that $u_p(x)$ and $u_q(x)$ are mutually orthogonal with respect to the weight function $w(x)$. Generally, we write:

$$\int_a^b w(x) u_p(x)\, u_q(x) = \begin{cases} 0 & p \neq q \\ \int_a^b w(x) u^2{}_p(x) dx & p = q, \end{cases} \qquad (14)$$

Definition 8.

We defined absolute error as:

Absolute Error $= |Y(x) - y_n(x)|; \quad 0 \leq x \leq 1, \qquad (15)$

where $Y(x)$ is the exact solution and $y_n(x)$ is the approximate solution.

MATERIALS AND METHODS

In this section, we constructed our orthogonal polynomials using the general weight function of the form: $w(x) = (a + bx^i)^k$.

This corresponds to quartic functions for $a = 1, b = -1, k = 1$, and $i = 4$, respectively, satisfying the orthogonality conditions in the interval $[a, b]$ under consideration.

According to Gram-Schmidt orthogonalization process, the orthogonal polynomial $u_j(x)$ Valid in the interval [a,b] with the leading term x^j, is given as:

$$u_j(x) = x^j - \sum_{i=0}^{j-1} a_{j,i} u_i(x) \qquad i = 0,1,2 \dots\dots j-1$$
and $j \geq 1 \qquad (16)$

Where $u_j(x)$ is an increasing polynomial of degree j and $u_i(x)$ are the corresponding values of the approximating functions in x. Then, starting with $u_0(x)$ = 1, we find that the linear polynomial $u_j(x)$ with leading term x, is written as:

$$u_1(x) = x + a_{1,0} u_0(x) \qquad (17)$$

Where $a_{1,0}$ is a constant to be determined. Since $u_1(x)$ and $u_0(x)$ are orthogonal, we have:

$$\int_a^b w(x) u_1(x) u_0(x)\, dx = 0 = \int_a^b x w(x) u_0(x)\, dx + a_{1,0} \int_a^b w(x) u^2{}_0(x)\, dx \qquad (18)$$

Using (14) and (18). From the above, we have:

$$a_{1,0} = \frac{\int_a^b w(x) x u_0(x)}{\int_a^b w(x) u^2{}_0(x)\,dx} \qquad (19)$$

Hence, substituting (19) into (16) gives:

$$u_1(x) = x + \frac{\int_a^b w(x) x u_0(x)}{\int_a^b w(x) u^2{}_0(x)\,dx} \qquad (20)$$

Proceeding in this way, the method is generalized and is written as:

$$u_j(x) = x^j + a_{j,0}u_0(x) + a_{j,1}u_1(x) + a_{j,2}u_2(x) + \cdots a_{j,j-1}u_{j-1}(x) \qquad (21)$$

Where the constants $a_{j,0}$ are so chosen such that $u_j(x)$ is orthogonal to $u_0(x), u_1(x), u_2(x) \ldots.. u_{j-1}(x)$. These conditions yield:

$$a_{j,i} = -\frac{\int_a^b x^j w(x) u_0(x)}{\int_a^b w(x) u^2{}_0(x)\,dx} \qquad (22)$$

For $k = 1, a = 1, b = -1$ and $i = 4$ valid in $[0,1]$

$$w(x) = 1 - x^4 \qquad (23)$$

$$u_0(x) = 1 \qquad (24)$$

We have $\mathrm{k} = 1, \mathrm{j} = 1$ and $u_0(x) = 1$, we write equation (16) as:

$$u_1(x) = x - a_{1,0}u_0(x) \qquad (25)$$

Simplifying the above equation, we have:

$$u_1(x) = x, u_2(x) = x^2 - \frac{5}{21} \qquad (26)$$

The shifted equivalent of the (26) that is valid in $[0,1]$ are given as:

$$u_0{}^*(x) = 1,$$
$$u_1{}^*(x) = 2x - 1, u_2{}^*(x) = 4t^2 - 4x + \frac{16}{21} \qquad (27)$$

In this work the method assumed an approximate solution with the orthogonal polynomial as basis function as:

$$u(x) \cong u_n(x) = \sum_{i=0}^{n} a_i u_i{}^*(x) \qquad (28)$$

Where $u_i{}^*(x)$ denotes the orthogonal polynomial of degree N where a_i, $\mathrm{i} = 0,1,2,\ldots$ are constants.

Standard Least Squares Method (SLSM)

In order to obtain the numerical solution of the Fractional Integro-Differential Equation of type, the usual least square approach with orthogonal polynomials created as the basis function is used. This technique is based on approximating the unknown function u(x) by assuming an approximation solution of the form stated. We consider equation (1) operating with $J^{\propto}$ on both sides as follows below.

The standard least squares method with the constructed orthogonal polynomials as basic function is applied to find the numerical solution of fractional Integro-differential equation of the type in (1) and (2). This method is based on approximating the unknown function $u(x)$ by assuming an approximation solution of the form defined in (28).

Consider equation (2) operating with $J^{\propto}$ on both sides as follows:

$$J^{\propto} D^{\alpha} u(x) = J^{\propto} f(x) + J^{\propto}\left(\int_0^1 k(x,t) u(t)\,dt\right) \qquad (29)$$

$$u(x) = \sum_{k=0}^{m-1} u^k(0^+)\frac{x^k}{k!} + J^{\propto} f(x) + J^{\propto}\left[\int_0^1 k(x,t) u(t)\,dt\right] \qquad (30)$$

$$\sum_{i=0}^{n} a_i u_i{}^*(x) = \sum_{k=0}^{m-1} u^k(0^+)\frac{x^k}{k!} + J^{\propto} f(x) + J^{\propto}\left[\int_0^1 k(x,t) \sum_{i=0}^{n} a_i u_i{}^*(t)\,dt\right] \qquad (30.b)$$

Hence, the residual equation is obtained as:

$$R(a_0, a_1, \dots \dots \dots, a_n) = \sum_{i=0}^{n} a_i u_i^*(x) - \{\sum_{k=0}^{m-1} u^k(0^+)\frac{x^k}{k!} + J^{\propto} f(x) + J^{\propto}[\int_0^1 k(x,t)\sum_{i=0}^{n} a_i u_i^*(t)\, dt]\} \quad (31)$$

Let,

$$S(a_0, a_1, \dots \dots \dots, a_n) = \int_0^1 [R(a_0, a_1, \dots \dots \dots, a_n)]^2 w(x) dx \quad (32)$$

Where $w(x)$ is the positive weight function defined in the interval, $[a, b]$. In this work, we take $w(x) = 1$ for simplicity. Thus:

$$S(a_0, a_1, \dots \dots \dots, a_n) = \int_0^1 \left\{\sum_{i=0}^{n} a_i u_i^*(x) - \{\sum_{k=0}^{m-1} u^k(0^+)\frac{x^k}{k!} + J^{\propto} f(x) + [\int_0^1 k(x,t)\sum_{i=0}^{n} a_i u_i^*(t)\, dt]\}\right\}^2 dx \quad (33)$$

In order to minimize the equation, we obtained the values of $a_i (i \geq 0)$ by finding the minimum value of S as:

$$\frac{\partial S}{\partial a_i} = 0, i = 0,1,2 \dots, n \quad (34)$$

Applying (34) on (33), we have:

$$\int_0^1 \left\{\sum_{i=0}^{n} a_i u_i^*(x) - \{\sum_{k=0}^{m-1} u^k(0^+)\frac{x^k}{k!} + J^{\propto} f(x) + J^{\propto}[\int_0^1 k(x,t)\sum_{i=0}^{n} a_i u_i^*(t)\, dt]\}\right\} dx \times \int_0^1 \left\{u_i^*(x) - J^{\propto}(\int_0^1 k(x,t) u_i^*(t) dt)\right\} dx \quad (35)$$

Thus, (36) are then simplified for $i = 0,1, \dots n$ to obtain $(n+1)$ algebraic system of equations in $(n+1)$ unknown a'_i s which are put in matrix form as follow:

$$A = \begin{pmatrix} \int_0^1 R(x,a_0)h_0 dx & \int_0^1 R(x,a_1)h_0 dx \cdots & \int_0^1 R(x,a_n)h_0 dx \\ \int_0^1 R(x,a_0)h_1 dx & \int_0^1 R(x,a_1)h_1 dx \cdots & \int_0^1 R(x,a_n)h_1 dx \\ \vdots & \vdots \quad \ddots & \vdots \\ \int_0^1 R(x,a_0)h_n dx & \int_0^1 R(x,a_1)h_n dx \dots & \int_0^1 R(x,a_n)h_n dx \end{pmatrix},$$

$$B = \begin{pmatrix} \int_0^1 \left[J^{\propto} f(x) + \sum_{k=0}^{m-1} u^k(0^+)\frac{x^k}{k!}\right] h_0 dx \\ \int_0^1 \left[J^{\propto} f(x) + \sum_{k=0}^{m-1} u^k(0^+)\frac{x^k}{k!}\right] h_1 dx \\ \vdots \\ \int_0^1 \left[J^{\propto} f(x) + \sum_{k=0}^{m-1} u^k(0^+)\frac{x^k}{k!}\right] h_n dx \end{pmatrix} \quad (36)$$

Where

$$h_i = u_i^*(x) - J^{\propto}[\int_0^1 k(x,t) u_i^*(t) dt], i = 0,1, \dots, n \quad (37)$$

$$R(x, a_i) = \sum_{i=0}^{n} a_i u_i^*(x) - J^{\propto}[\int_0^1 k(x,t)\sum_{i=0}^{n} a_i u_i^*(t)\, dt], i = 0,1, \dots, n \quad (38)$$

The $(n+1)$ linear equation are then solved using Gaussian elimination method or any suitable computer package like maple 18 to obtain the unknown constants $a_i(i = 0(1)n)$, which are then substituted back into the assumed approximate solution to give the required approximation solution.

Example 1: Consider the following fractional Integra-differential (Khosrow *et al*., 2013):

$$D^{\frac{3}{4}}u(x) = -\frac{x^2 e^x}{5}u(x) + \frac{6x^{2.25}}{\Gamma(3.25)} + e^x \int_0^x tu(t)dt \tag{39}$$

Subject to $u(0) = 0$. The exact solution is:

$$u(x) = x^3 \tag{40}$$

Applying the above method on (39) to have the required approximate solution as:

$$u(x) = 3 \times 10^{-11} + x^3 \tag{41}$$

Example 2: Consider the following fractional Integra-differential (Mohamed et al., 2016):

$$D^{\frac{1}{2}}u(x) = u(x) + \frac{8x^{2.25}}{3\Gamma(0.5)} - x^2 - \frac{1}{2}x^3 + \int_0^x tu(t)dt \tag{42}$$

Subject to $u(0) = 0$. The exact solution is:

$$U(x) = x^2 \tag{43}$$

Applying the above method on (42) to have the required approximate solution as:

$$u(x) = 6.428498356 \times 10^{-8} + 3.20 \times 10^{-7}x^2 + 5.057252466 \times 10^{-7}x^3 \tag{44}$$

RESULTS

Tables of Results

Table 1: Numerical Results of Example 1.

x	Exact Solution	Approximate Solution	Absolute Error
0.0	0.000	0.00000000003000	3.000E-11
0.1	0.001	0.00100000003000	3.000E-11
0.2	0.005	0.006600000003000	3.000E-11
0.3	0.288	0.03200000003000	3.000E-11
0.4	0.0633	0.06900000003000	3.000E-11
0.5	0.127	0.22500000000000	0.00E+00
0.6	0.200	0.21500000000000	0.00E+00
0.7	0.377	0.34500000000000	0.00E+00
0.8	0.511	0.54500000000000	0.00E+00
0.9	0.99	0.72800000000000	0.00E+00
1.0	1.000	1.00000000000000	0.00E+00

Table 2: Numerical Results of Example 2.

x	Exact Solution	Approximate Solution	Absolute Error
0.0	0.00	0.00000006428498	6.428E-08
0.1	0.011	0.01000009461000	9.460E-08
0.2	0.041	0.04000012359000	1.236E-07
0.3	0.090	0.09000015426000	1.543E-07
0.4	0.161	0.16000018970000	1.897E-07
0.5	0.252	0.25000023290000	2.329E-07
0.6	0.366	0.36000028680000	2.868E-07
0.7	0.492	0.49000035470000	3.546E-07
0.8	0.641	0.64000043930000	4.393E-07
0.9	0.811	0.81000054390000	5.439E-07
1.0	1.000	1.00000067200000	6.714E-07

DISCUSSION OF RESULTS

All the two numerical examples presented in this study were solved using MAPLE 18. The Tables of error for the examples shows that the method with the constructed orthogonal polynomials is accurate and converges at the lower numbers of the approximate.

CONCLUSION

The study showed that the method with the constructed orthogonal polynomials is successfully used for solving FIDEs in a wide range with three examples. The method gives more realistic series solutions that converge very rapidly in fractional equations. The results obtained showed that the method is powerful when compared with the exact solutions and also show that there is a similarity between the exact and the approximate solution.

Calculation showed that SLSM is a powerful and efficient technique to find a very good solution for this type of equation as well as analytical solutions to numerous physical problems in science and engineering. Also, the results presented in Tables 1 and 2 further demonstrate of the method.

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