Mathematical Model for the Dynamics of Drug-Abuse and Violence Co-Menace

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ABSTRACT

In this work, we propose a drug-abuse and drug-induced violence co-menace epidemiological model using deterministic non-linear system of differential equations. We consider drug-abuse as well as violence sub-models. The vice(s)-free, endemic steady states were obtained in order to obtain basic reproduction numbers , analyze local and global stability at the vice(s)-free steady states for the model, and carried out numerical simulation for the sub-models and the co-menace model.

We conclude that a given population faces the problem of drug-abuse as a result of huge number of individuals going into the menace of low drug-abuse more than the sum rates at which individuals leave low drug-abuse class through individuals' progression from low to high drug-abusing, low drug-abusers going for rehabilitation, natural death and drug-induced death. Similarly, a given population is subjected to the problem of violence for the rate at which individuals become violent is more than the rate at which individuals leave violence class through movement to victims class, movement to rehabilitation center, natural death and violence-induced death.

We recommend that much more effort needs to be put in place to prevent people from joining low drug-abuse and eliminate or drastically reduce the number of low drug-abusers to be less than the total sum of the individuals leaving the low drug-abuse class. We also recommend that greater efforts are needed to prevent people from becoming violent and eliminate or drastically reduce the number of violent persons to be less than the total sum of the individuals leaving the violence class.

(Keywords: mathematical model, drug abuse, violence, co-menace, mathematical model, basic reproduction number)

INTRODUCTION

The use of drugs to cure an illness, prevent a disease or improve health is termed drug use. But when a drug is taken for reasons other than medical, in an amount, strength, frequency or manner that causes damage to the physical or mental functioning of an individual, it becomes drug-abuse (National Institute on Drug-abuse 2014). Drug-abuse is at times called substance abuse.

In 1956, World Health Organization (WHO) and American Psychological Association (APA) considered drug-abuse as a disease which is defined by the illicit consumption of any naturally occurring or pharmaceutical substance for the purpose of changing the way, in which a person feels, thinks, or behaves without understanding or taking into consideration the damaging physical and mental side-effects that are caused (Kumar and Sahu, 2016).

It was reported that marijuana was the most widely consumed drug in 2016 with 192 million people using it at least once during the previous year (United Nations Office on Drug and Crime, 2017). Marijuana was found to be the most commonly abused drug in Nigeria, where as 4.6 million people were found to be using tramadol and cough syrups for non-medical purpose in 2018, which positioned Nigeria as second worldwide in that regard (Ayodamola, 2019).

Violence can be regarded as any form of unwanted behavior or action which causes physical or psychological discomfort, threat, harm, deprivations, or even death to one-self or other individual(s) as victim(s). Violence could be domestic or otherwise and could either be carried out individually or collectively. Whether violence is carried out individually or collectively, it takes many forms such as physical violence, spiritual violence, psychological violence, structural violence, emotional violence, sexual violence, religious violence, cultural violence, tribal violence, communal violence, verbal abuse, financial abuse and neglect among others (WHO, 2018).

There is no doubt that drug-abuse helps tremendously in bringing about the evolution of local, national, and international terror groups across the globe. For instance, the emergence of *Sheela* Boys in Adamawa state, *Daba* or *Jagaliya* Boys in Kano state, *Kalare* Boys in Gombe state, *Sara-Suka* Boys in Bauchi state, Area Boys in Lagos state, Militant Group in Delta state, *Boko-Haram* Insurgent Group, *Taliban*, Revolutionary Armed Forces of Colombia (FARC), etc. Moreover, the terrible operations of the evolved terror groups endanger the survivability of the entire public.

Petty crimes such as bag wrestling, phone snatching, money wrestling, organized crimes such as armed robbery, terrorism, kidnapping, militancy, vandalism, piracy, cattle rustling, mob action, rape, extra-judicial killing by security agents, theft, burglary, crises, ritualism, cultism, communal fight and tribal clashes, which several parts of the world are plunged into, are some of the numerous devastating problems in most cases associated with drug-abuse influence.

There is a documented fact which reveals Taliban involvement in trade of illicit drug (opioid) in Afghanistan. Also, there is evidence of involvement of the FARC in Colombia in the coca/cocaine illicit trade, before the peace agreement of 2016. Income from drugs is key for some groups and only one revenue stream of many for most terrorist groups, (UN Office on Drug and Crime 2017).

Violence is defined as the use of physical force, power or weapon against one-self, another person, a group or community that either results in, or has a high likelihood of resulting in threat or psychological harm, deprivations, injury or death (WHO, 2002). It is reported that over 1.6 million people aged between 15 – 44 years loss their lives to violence every year, with 14% as males and 7% as females (WHO, 2018). Report reveals that about 129 million people consume marijuana in 2016 (UN Office on Drug and Crime 2017). Similarly, there were 4.6 million Nigerians abusing tramadol in 2018 as estimated (Ayodamola, 2019).

Over 1.6 million people aged between 15 – 44 years, loss their lives to violence every year with 14% males and 7% females (WHO, 2018). Similarly, there were 22,887 deaths in Nigeria in 2014 alone and there were 2,182 violent deaths in 2017 alone in Nigeria as estimated (Nigeria Watch, 2017).

Many efforts have been put in place to model the dynamics of drug-abuse in populations, some of which are: White and Comiskey (2007) proposed a model which described heroin use within a fixed community where they considered individuals as divided into Susceptible, Drug Users and Drug Users Undergoing Treatment. They showed the existence of a stable state equilibrium and suggested both a situation where heroin use will be eradicated and one where it will remain an endemic.

Bobashev, Holloway, Solano and Gutkin (2017) proposed a heuristic control theory model that describes smoking under restricted and unrestricted access to cigarettes. The model was based on the allostasis theory and uses a formal representation of a multi-scale opponent process.

The model stimulates smoking behavior of an individual and produces both short-term and long term smoking patterns. By introducing a formal representation of withdrawal and craving-like processes, the model produces gradual increases over time in withdrawal and craving-like signals associated with abstinence and shows that after three (3) months of abstinence, craving disappears.

The model was programmed as a computer application allowing users to select simulation scenarios. The application links images of brain regions that are activated during the binge/intoxication, withdrawal, or craving with corresponding simulated states.

The model was calibrated to represent smoking patterns described in peer-reviewed literature; however, it is generic enough to be adapted to other drugs, including cocaine and opioids. Although the model does not mechanistically describe specific neurobiological processes, it can be useful in prevention and treatment practices as an illustration of drug-using behaviors and expected dynamics of withdrawal and craving during abstinence.

Rossi (2002) proposed a model to reflect the spread of drug use in population and enumerated the role of dynamic modeling in drug-abuse epidemiology. The researcher stated that dynamic models can be used to generate estimates where data are sparse or to verify hypotheses or predict trends by means of "what if" scenario analyses. He also enumerated that the models that can be used effectively in the drug field are essentially models of epidemics that describe the spread of a disease in a population in order to provide evidence for public health-oriented interventions and policies.

Santoro, Triolo, and Rossi (2013) proposed a six-typology compartment model of trends in the use of illicit drugs in Italy. A deterministic linear ODE framework, where parameters were estimated using Italian data was obtained and analyzed. The system showed an evolution toward a steady state. The model represented a theoretical development in drug policy analysis, as it shows the relevance of flux parameters, which were in principle subject to modifications.

Mushayabasa and Tapedzesa (2015) presented a new mathematical modeling framework to investigate the effects of illicit drug use in community. The transmission process in the model was captured as a social "contact" process between the susceptible individuals and illicit drug users.

Epidemic and endemic analysis were conducted with a focus on the threshold dynamics characterized by the basic reproduction number. Illustrative numerical results were presented with a case study in Cape Town, Gauteng, Mpumalanga and Durban communities of South Africa. The model was extended to incorporate time dependent intervention strategies.

Princess, Nyabadza, Landi and Hui (2016) considered a compartmental model of substance users in the two forms of rehabilitation. A periodic function was used to illustrate fluctuations of drug users entering rehabilitation. Two basic reproduction numbers $\Re_0$ and $[\Re_0]$ where $\Re_0$ is for model with fluctuations and $[\Re_0]$ for the model without.

The model was analyzed in terms of the two reproduction numbers. The model was fitted to data on the two forms of rehabilitations and projections made. The main results showed that the disease goes to extinction if the threshold value $\Re_0$ is less than unity, whilst the disease persists if the threshold value is larger than unity. Also, there exists a positive solution when the threshold is above unity.

Sebil and Otoo (2014) used a simple continuous model for the spread of domestic violence in Tamale metropolis, using ordinary differential equations.

A modeling technique of Abusive, Susceptible and Violence Victims (ASV), similar to the Susceptible, Infectious and Recovered (SIR) model in epidemics, was used for formulating the spread of domestic violence as a system of differential equations. The system of differential equations was analyzed by linearization of nonlinear systems and non-dimentionalization to predict the behavior of the spread of Domestic Violence and found to be exponential at early stage. Hove and Gwazane (2011) investigated factors associated with domestic violence amongst four couple groups namely: concordant negative, concordant positive, discordant male positive and discordant female positive.

Data from the Zimbabwe Demographic Health survey (2005-2006) was used to run a Chi-square test to compare couples on demographic factors, economic factors and decision making. A logistic regression model which helps to control for confounding was fitted to identify factors associated with domestic violence for the different couple groupings.

It was found out that, concordant positive couples experienced the most domestic violence with 53.5% reporting that they experienced domestic violence and discordant female positive couples experienced the least domestic violence with 39.4% of them reporting that they experienced domestic violence.

It was also found out that, among the factors associated with domestic violence, wealth has an inverse relationship with domestic violence. They concluded that higher household economic status was associated with less domestic violence in marriage and thus we recommended that communities engage in income generating projects as an indirect way of reducing domestic violence.

Sooknanan, Bhatt and Comissiong (2013) studied criminal gang membership treated as an infection that spreads through a community by interactions among gang members and a population. A system of coupled nonlinear ordinary differential equations was used to describe this spread. Three equilibrium states were obtained, two of which contain no gang members. Sensitivity analysis on parameters such as recruitment, conviction and recidivism rates and longer jail sentences were varied and it was evident that the greatest reduction occurs by ganging the parameters in combination. A bifurcation analysis was carried out and it shows transcritical bifurcation and no Hopf bifurcation.

Mondal and Mondal (2017) modified a model by Nuno*et al*. They modified the interaction function between criminals and guards and investigated the impacts of the modification and modified the number of arrests per unit time. The steady state solutions of the proposed model were determined and their stability natures were discussed. They carried out numerical simulation results to verify the analytical stability conditions.

Lawrence, Elisha, Anne, and Lawi (2018) proposed a more detailed diffusion type differential equations model that describes how the number of criminals evolved in a specific area. Their model allows for two distinct criminal types associated with major and minor crime. They also examined a stochastic variant of the model that represents more realistically the 'generation' of new criminals. Numerical solutions from both models were presented and compared with actual crime data for the greater Manchester area and agreement with the actual data was observed. A preliminary statistical analysis of the data also supports the model's potential to describe crime.

McMillon, Simon and Morenoff (2014) built and analyzed a series of dynamical systems models of the spread of crime, imprisonment and recidivism, using only abstract transition parameters.

They computed analytic expressions for the equilibria and for the tipping points between high-crime and low-crime equilibria in the models to find the general patterns among these parameters and patterns that are independent of the underlying particulars. They used the expressions particularly to examine the effects of longer prison terms and of increased incarceration rates on the prevalence of crime, with a follow-up analysis on the effects of a Three-Strike Policy.

There have been many efforts to curtail the spread of drug-abuse as well as violence including drug-abuse model proposed by White and Comiskey (2007) and violence model proposed by Sebil and Otoo (2014) but there has not been a proposed model that exhibits the dynamics of the two social vices for study and control. Therefore, based on the fact from Atkinson, Anderson, Hughes, Bellis, Sumnall and Syed (2009) that, involvement in illicit drug use can increase the risks of being both a victim and/or perpetrator of violence, while experiencing violence can increase the risks of initiating illicit drug use; and the fact from National Institute on Drug-abuse (2014) that, there are different classes of drug-abusers depending on the quantity of drugs consumed such as drug-abusers and drug addicts therefore, we develop aco-menace mathematical model (based on SIR model) that exhibits the dynamism of the two social vices (as in co-infection models), classifying the drug-abusers as low and high, to study the dynamics of the two vices(drug-abuse and violence) and suggest possible control strategies to the problems.

Model Formulation

The model is governed by the following assumptions: The population under study comprises 8 epidemiological sub-classes as Susceptible individuals S, Low Drug-Abusers L_D, High Drug-Abusers H_D, Low Drug-Abusing and Violent individuals L_D^V, High Drug-Abusing and Violent individuals H_D^V, Violent individuals V ,Violence victims V_V and Rehabilitants R .

The population under study is homogeneously mixed. Both low and high drug-abusers persuade susceptible persons to become low drug-abusers. Violent persons are less bound to become drug-abusers as well; so do not become either low or high drug-abusers. The population under study is closed at the duration of the study. Recruitment into the population is by birth into the susceptible class. The rehabilitation center is in-patient so that relapsing effect is avoided. The rehabilitation center only provides temporary reformation to the drug-abusers, violent persons and those who are both drug-abusers and violent individuals.

The population is divided into eight compartments that comprises of Susceptible persons (S), Low Drug-abusers (L_D), High Drug-abusers (H_D), Violent individuals (V), Low Drug-abusing and Violent individuals (L_D^V), High Drug-abusing and Violent individuals (H_D^V), Violence victims (V_V) and individuals in Rehabilitation centers (R). Susceptible compartment increases by inflow of individuals through birth, at the rate, τ and inflow of individuals who were rehabilitated at a rate θ.

The Susceptible compartment reduces by $\frac{(\beta_1 L_D + \pi\beta_2 H_D)S}{N}$, where β_1 is the rate at which susceptible become a low drug-abusing individual while the rate at which susceptible become a high drug-abusing individual is represented by β_2 and π represent the modification parameter for individuals who became high drug-abusing individual.

The Susceptible is further reduced by $\frac{\beta_3 VS}{N}$, where β_3 is the rate at which a susceptible individual become a violent individual. Natural death rate for the whole population is represented by μ. The probability for a violent individual moving to rehabilitation is represented by ρ_3, while violent induced death rate is represented by δ_2.

Violent compartment is also reduced by $\frac{\beta_4 VS}{N}$, where β_4 is rate at which result to violent victim individual as a result of interaction between the violent individual with susceptible individual. Low drug abusing class decreases by the movement of individual to the low drug abuser and violent individuals' class by $\frac{\alpha_1 L_D V}{N}$, where α_1 is the rate at which a low drug-abuser became low drug-abuser and violent individual. While δ_1 represents the death rate of being low drug-abuser and violent individual. We assume that low drug-abuser will become high drug abuser at a rate ε_1, while, ρ_1 is the probability of low drug abusing persons going into rehabilitation.
The probability at which a high drug abuser going into rehabilitation is represented by ρ_2. The low drug-abuser and violent individual progresses to become high drug abuser and violent individual at a rate ε_1 while $\frac{\alpha_3 L_D^V S}{N}$ is the interaction between the susceptible and the low drug-abuser, where α_3 is the rate at which low drug abusing and violent individuals become violent victims.
The low drug abusing and violent induced death rate is represented by δ_1. $\frac{\alpha_2 H_D V}{N}$ is the movement of high drug abuser to high drug-abuser and violent, where α_2 is the rate at which high drug-abusers become high drug-abusers and violent and σ_2 represents the high drug abuse induced death rate. $\frac{\alpha_4 H_D^V S}{N}$ is the interaction between the susceptible and high drug abusing and violent individual whereas α_4 is the rate at which susceptible persons become violence victims for interacting with high drug abusing and violent individuals. δ_2 represent high drug abusing and violent individuals induced death rate. However, the violent victim compartment is reduced by the induced death at a rate δ_4. It is also assumed that individual in undergoing rehabilitation can die as a results vices in rehabilitation class at a rate σ_2.

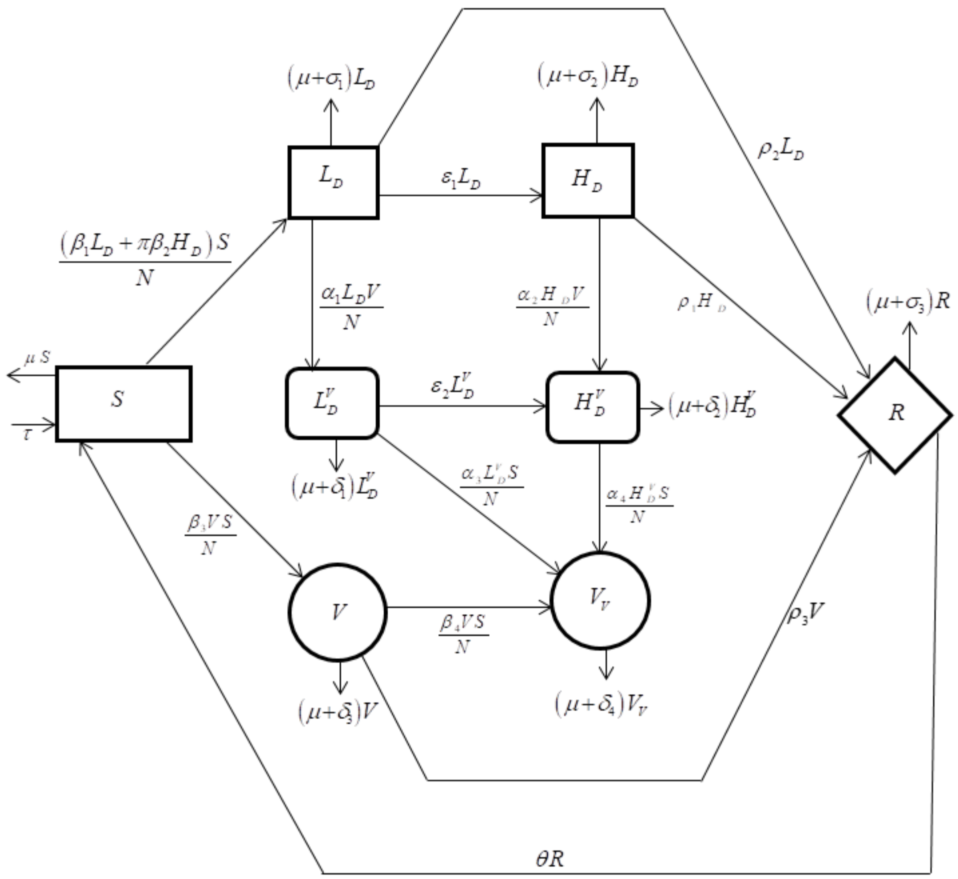

Figure 1: A Schematic Flow Diagram of Drug-Abuse and Violence (Co-Menace) Model.

The model equations are presented in (1).

$$\left.\begin{aligned}
\frac{dS}{dt} &= \tau+\theta R-\frac{\beta_1 L_D S}{N}-\frac{\pi\beta_2 H_D S}{N}-\frac{\beta_3 VS}{N}-\mu S\\
\frac{dL_D}{dt} &= \frac{\beta_1 L_D S}{N}+\frac{\pi\beta_2 H_D S}{N}-\frac{\alpha_1 L_D V}{N}-\varepsilon_1 L_D-\rho_2 L_D-(\mu+\sigma_1)L_D\\
\frac{dH_D}{dt} &= \varepsilon_1 L_D-\frac{\alpha_2 H_D V}{N}-\rho_1 H_D-(\mu+\sigma_2)H_D\\
\frac{dL_D^V}{dt} &= \frac{\alpha_1 L_D V}{N}-\frac{\alpha_3 L_D^V S}{N}-\varepsilon_2 L_D^V-\frac{\alpha_4 H_D^V S}{N}-(\mu+\delta_1)L_D^V\\
\frac{dH_D^V}{dt} &= \frac{\alpha_2 H_D V}{N}+\varepsilon_2 L_D^V-\frac{\alpha_4 H_D^V S}{N}-(\mu+\delta_2)H_D^V\\
\frac{dV}{dt} &= \frac{\beta_3 VS}{N}-\frac{\beta_4 VS}{N}-\rho_3 V-(\mu+\delta_3)V\\
\frac{dV_V}{dt} &= \frac{\alpha_3 L_D^V S}{N}+\frac{\alpha_4 H_D^V S}{N}+\frac{\beta_4 VS}{N}-(\mu+\delta_4)V_V\\
\frac{dR}{dt} &= \rho_1 H_D+\rho_2 L_D+\rho_3 V-\theta R-(\mu+\sigma_3)R
\end{aligned}\right\}$$

(1)

Where,

$$N(t)=S(t)+L_D(t)+H_D(t)+L_D^V(t)+H_D^V(t)+V(t)+V_V(t)+R(t)$$

(2)

MODEL ANALYSIS

Drug-Abuse Only Model

The drug-abuse only equation is given as:

$$\left.\begin{aligned}
\frac{dS}{dt} &= \tau+\theta R-\frac{\beta_1 L_D S}{N_D}-\frac{\pi\beta_2 H_D S}{N_D}-\mu S\\
\frac{dL_D}{dt} &= \frac{\beta_1 L_D S}{N_D}+\frac{\pi\beta_2 H_D S}{N_D}-\varepsilon_1 L_D-\rho_2 L_D-(\mu+\sigma_1)L_D\\
\frac{dH_D}{dt} &= \varepsilon_1 L_D-\rho_1 H_D-(\mu+\sigma_2)H_D\\
\frac{dR}{dt} &= \rho_1 H_D+\rho_2 L_D-\theta R-(\mu+\sigma_3)R
\end{aligned}\right\}$$

(3)

where

$$N_D(t)=S(t)+L_D(t)+H_D(t)+R(t) \qquad \textbf{(4)}$$

Stability Analysis of Drug-Abuse-Free Equilibrium

The drug abuse free equilibrium and endemic equilibrium are given as:

$$E_0^D = (S^*, L_D^*, H_D^*, R^*) = \left(\frac{\tau}{\mu}, 0, 0, 0\right) \qquad \textbf{(5)}$$

And

$$E_1^D = (S^*, L_D^*, H_D^*, R^*)$$

where

$$\left.\begin{aligned} S^* &= \frac{N^*\{\tau(\theta+\mu+\sigma_3)+\theta(\rho_3 H_D^* + \rho_2 L_D^*)\}}{(\beta_1 L_D^* + \pi\beta_2 H_D^* - \mu N^*)(\theta+\mu+\sigma_3)} \\ L_D^* &= \frac{\pi\beta_2 H_D^* A_1}{N_D^*(\varepsilon_1+\rho_2+\mu+\sigma_1)-\beta_1 A_1} \\ H_D^* &= \frac{\varepsilon_1 A_2}{(\rho_1+\mu+\sigma_2)} \\ R^* &= \frac{\rho_1 A_3 + \rho_2 A_2}{(\theta+\mu+\sigma_3)} \end{aligned}\right\} \qquad \textbf{(6)}$$

We calculate the reproduction number for the drug abuse-only model R_0^D as:

$$R_0^D = \frac{\beta_1}{(\varepsilon_1+\rho_2+\mu+\sigma_1)} + \frac{\pi\beta_2\varepsilon_1}{(\varepsilon_1+\rho_2+\mu+\sigma_1)(\rho_1+\mu+\sigma_2)} \qquad \textbf{(7)}$$

The drug-abuse free equilibrium E_0^D point of drug-abuse model is locally asymptotically stable if $R_0^D < 1$ and unstable if $R_0^D > 1$.

Global Stability of Drug-Abuse Only Model

Theorem 3: A fixed point $U_0 = \left(\frac{\tau}{\mu}, 0\right)$ is a global asymptotic stable equilibrium point of the system if and only if the condition $\beta_1 L_D = \pi\beta_2 H_D$ holds.

Proof: Let the drug-abuse sub-system (3) be written as:

$$(H1): \frac{dX}{dt} = T(X, I)$$

$$(H2): \frac{dI}{dt} = G(X, I), G(X,0) = 0,$$

where $X = (S, R) \in \Re^2$, $I = (L_D, H_D) \in \Re^2$,

$T(X,0) = \left(\frac{\tau}{\mu}, 0\right) = E_0^D$ and

$$A = D_I G(X^*, 0) = \begin{bmatrix} \beta_1 - (\varepsilon_1+\rho_2+\mu+\sigma_1) & \pi\beta_2 \\ \varepsilon_1 & -(\rho_1+\mu+\sigma_2) \end{bmatrix}$$

The off diagonal entries of A are non-negative.

$$\hat{G}(X, I) = \begin{bmatrix} \hat{G}_1(X, I) \\ \hat{G}_2(X, I) \end{bmatrix} = \begin{bmatrix} \beta_1 L_D (1 - \frac{S}{N_D}) - \frac{\pi\beta_2 H_D S}{N_D} \\ 0 \end{bmatrix}$$

Since $0 \le S \le N_D$ and $\beta_1 L_D < \pi\beta_2 H_D$, $\hat{G}_1(X, I) < 0$, $\hat{G}_2(X, I) = 0$, which obviously shows that in the second condition $(H2)$, $\hat{G}(X, I) \ge 0$ is not satisfied. However, the drug-abuse free equilibrium point is globally asymptotically stable if $\beta_1 L_D \left(1 - \frac{S}{N_D}\right) \ge \frac{\pi\beta_2 H_D S}{N_D}$.

Violence Only Model

$$\left.\begin{aligned} \frac{dS}{dt} &= \tau + \theta R - \frac{\beta_3 VS}{N_V} - \mu S \\ \frac{dV}{dt} &= \frac{\beta_3 VS}{N_V} - \frac{\beta_4 VS}{N_V} - \rho_3 V - (\mu+\delta_3)V \\ \frac{dV_V}{dt} &= \frac{\beta_4 VS}{N_V} - (\mu+\delta_4)V_V \\ \frac{dR}{dt} &= \rho_3 V - \theta R - (\mu+\sigma_3)R \end{aligned}\right\} \qquad \textbf{(8)}$$

$$N_V(t) = S(t) + V(t) + V_V(t) + R(t) \qquad \textbf{(9)}$$

Stability Analysis of Violence-Free Equilibrium Point

The violence free equilibrium and endemic equilibrium are given as:

$$E_0^V = (S^*, V^*, V_V^*, R^*) = \left(\frac{\tau}{\mu}, 0, 0, 0\right) \qquad \textbf{(10)}$$

and

$$E_0^V = (S^*, V^*, V_V^*, R^*)$$

where

$$\left.\begin{aligned}
S^* &= \frac{N_V^*(\rho_3+\mu+\delta_3)}{(\beta_3-\beta_4)}\\
V^* &= \frac{N_V^*(\theta+\mu+\sigma_3)(\tau+\mu)}{A_1\beta_3(\theta+\mu+\sigma_3)-N_V^*\theta\rho_3}\\
V_V^* &= \frac{\beta_4 A_5 A_4}{N_V^*\left(\mu+\delta_4\right)}\\
R^* &= \frac{\rho_3 A_5}{\left(\theta+\mu+\sigma_3\right)}
\end{aligned}\right\} \qquad \textbf{(11)}$$

Similarly, we calculate the reproduction number for violence only model as:

$$R_0^V = \frac{\beta_3}{(\rho_3+\mu+\delta_3)} \qquad \textbf{(12)}$$

The drug-abuse free equilibrium E_0^V point of drug-abuse model is locally asymptotically stable if $R_0^V < 1$ and unstable if $R_0^V > 1$.

Global Stability of the Violence Only Model

Theorem 6: A fixed point $U_0 = \left(\frac{\tau}{\mu}, 0\right)$ is a global asymptotic stable equilibrium point of the system if and only if the condition $\frac{\beta_4 VS}{N_V} \geq 0$ is satisfied.

Proof: Let the violence sub-system (8) be written as:

$$(H1): \frac{dX}{dt} = T(X,I)$$
$$(H2): \frac{dI}{dt} = G(X,I), G(X,0) = 0,$$

where $X = (S,R) \in \Re^2$, $I = (V, V_V) \in \Re^2$, $T(X,0) = \left(\frac{\tau}{\mu}, 0\right) = E_0^V$ and

$$A = \begin{bmatrix} \beta_3 - (\beta_4+\rho_3+\mu+\delta_3) & 0 \\ \beta_4 & -(\mu+\delta_4) \end{bmatrix}$$

which the off diagonal entries are non-negative.

Now $\hat{G}(X,I) = \begin{bmatrix} \hat{G}_1(X,I) \\ \hat{G}_2(X,I) \end{bmatrix} = \begin{bmatrix} \beta_3 V(1-\frac{S}{N_V}) + \frac{\beta_4 VS}{N_V} \\ -\frac{\beta_4 VS}{N_V} \end{bmatrix}$

Since $0 \leq S \leq N_V$, $\hat{G}_1(X,I) \geq 0$ but $\hat{G}_2(X,I) < 0$. Hence, the violence-free equilibrium state is globally asymptotically stable if $\frac{\beta_4 VS}{N_V} \geq 0$.

Drug Abuse-Violence Co-Menace Model

Recall that the model equations for the co-menace model is presented in (1).

Stability of Co-Menace-Free Equilibrium Point

The drug abuse-violence co-menace free equilibrium and endemic equilibrium are given as:

$$E_0^{VD} = \left(S^*, L_D^*, H_D^*, L_D^{V*}, H_D^{V*}, V^*, V_V^*, R^*\right) = \left(\frac{\tau}{\mu}, 0,0,0,0,0,0,0\right) \qquad \textbf{(13)}$$

and

$$E_1^{VD} = \left(S^*, L_D^*, H_D^*, L_D^{V*}, H_D^{V*}, V^*, V_V^*, R^*\right)$$

where

$$\left.\begin{aligned}
S^* &= \frac{N^*\left\{\tau(\theta+\mu+\sigma_3)+\theta(\rho_1 H_D^*+\rho_2 L_D^*+\rho_3 V)\right\}}{(\theta+\mu+\sigma_3)(\beta_1 L_D^*+\pi\beta_2 H_D^*+\rho_3 V+\mu N^*)}\\
L_D^* &= \frac{\pi\beta_2 H_D^* A_7}{(N^*(\varepsilon_1+\rho_2+\mu+\sigma_1)+\alpha_1 V^*-\beta_1 A_7)}\\
H_D^* &= \frac{N^*\varepsilon_1 A_8}{N^*(\rho_1+\mu+\sigma_2)+\alpha_2 V^*}\\
L_D^{V*} &= \frac{\alpha_1 A_8 V^*}{N^*(\varepsilon_2+\mu+\delta_1)+\alpha_3 A_7}\\
H_D^{V*} &= \frac{\alpha_2 A_9 V^*+N^*\varepsilon_2 A_{10}}{N^*(\mu+\delta_2)+\alpha_4 A_7}\\
V^* &= \frac{\left\{\tau(\theta+\mu+\delta_3)+\theta(\rho_1 A_9+\rho_2 A_8)\right\}(\beta_3-\beta_4)(\rho_3+\mu+\delta_3)(\rho_3+\mu+\delta_3)(\rho_3+\mu+\delta_3)(\beta_1 A_8+\pi\beta_2 A_9-\mu N^*)}{\left\{\rho_3(\rho_3+\mu+\delta_3)(\theta+\mu+\sigma_3)-\theta\rho_3(\beta_3-\beta_4)\right\}}\\
V_V^* &= \frac{\alpha_3 A_{10} A_7+\alpha_4 A_{11} A_7+\beta_4 A_{12} A_7}{N^*\left(\mu+\delta_4\right)}\\
R^* &= \frac{\rho_1 A_9+\rho_2 A_8+\rho_3 A_{12}}{(\theta+\mu+\sigma_3)}
\end{aligned}\right\} \qquad \textbf{(14)}$$

Similarly, we calculated the basic reproduction number for co-menace model as:

$$R_0^{VD} = \left\{ \frac{\left(b\beta_1 + \pi\beta_2\varepsilon_1\right)}{ab}, \frac{\beta_3}{e} \right\} \quad \textbf{(15)}$$

where

$a = \left(\varepsilon_1 + \rho_2 + \mu + \sigma_1\right)$, $b = \left(\rho_1 + \mu + \sigma_2\right)$
and $e = \left(\rho_3 + \mu + \delta_3\right)$.

The drug-abuse free equilibrium E_0^{VD} point of drug-abuse model is locally asymptotically stable if $R_0^{VD} < 1$ and unstable if $R_0^{VD} > 1$.

Global Stability of the Co-Menace Model

Theorem 8: A fixed point $U_0 = \left(\frac{\tau}{\mu}, 0\right)$ is a global asymptotic stable equilibrium point of the system ((1) if and only if the following conditions are satisfied:

(i) $\frac{\alpha_3 L_D^V S}{N} \geq \frac{\alpha_1 L_D V}{N}$,

(ii) $\frac{\alpha_4 H_D^V S}{N} \geq \frac{\alpha_2 H_D^V}{N}$,

(iii) $\left(\frac{\alpha_3 L_D^V S}{N} + \frac{\alpha_4 H_D^V S}{N} + \frac{\beta_4 VS}{N}\right) \geq 0$.

Proof: The co-menace system (1) can be written as:

$$\left.\begin{aligned} \frac{dX}{dt} &= T(X,I) \\ \frac{dI}{dt} &= G(X,I), G(X,0) = 0 \end{aligned}\right\} \quad \textbf{(16)}$$

where

$X = (S,R) \in \Re^2$, $I = (L_D, H_D, L_D^V, H_D^V, V, V_V) \in \Re^6$, $T(X,0) = \left(\frac{\tau}{\mu}, 0\right) = E_0^{VD}$ and

$$A = \begin{pmatrix} n_1 & \pi\beta_2 & 0 & 0 & 0 & 0 \\ \varepsilon_1 & n_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_3 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & n_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_5 & 0 \\ 0 & 0 & \alpha_3 & \alpha_4 & \beta_4 & n_6 \end{pmatrix}$$

where $n_1 = \beta_1 - (\varepsilon_1 + \rho_2 + \mu + \sigma_1)$, $n_2 = -(\rho_1 + \mu + \sigma_2)$, $n_3 = -(\alpha_3 + \varepsilon_2 + \mu + \delta_1)$, $n_4 = -(\alpha_4 + \mu + \delta_2)$, $n_5 = \beta_3 - (\beta_4 + \rho_3 + \mu + \delta_3)$ and $n_6 = -(\mu + \delta_4)$, which the off diagonal entries are nonnegative.

$$\hat{G}(X,I) = \begin{bmatrix} \hat{G}_1(X,I) \\ \hat{G}_2(X,I) \\ \hat{G}_3(X,I) \\ \hat{G}_4(X,I) \\ \hat{G}_5(X,I) \\ \hat{G}_6(X,I) \end{bmatrix} = \begin{bmatrix} (\beta_1 L_D + \pi\beta_2 H_D)(1 - \frac{S}{N}) + \frac{\alpha_1 L_D V}{N} \\ \frac{\alpha_2 H_D V}{N} \\ \frac{\alpha_3 L_D^V S}{N} - \frac{\alpha_1 L_D V}{N} \\ \frac{\alpha_4 H_D^V S}{N} - \frac{\alpha_2 H_D^V}{N} \\ \beta_3 V(1 - \frac{S}{N}) + \frac{\beta_4 VS}{N} \\ -\left(\frac{\alpha_3 L_D^V S}{N} + \frac{\alpha_4 H_D^V S}{N} + \frac{\beta_4 VS}{N}\right) \end{bmatrix}$$

Since $0 \leq S \leq N$ hence, the co-menace-free equilibrium state of the model is globally asymptotically stable if the following inequalities hold:

(i) $\frac{\alpha_3 L_D^V S}{N} \geq \frac{\alpha_1 L_D V}{N}$,

(ii) $\frac{\alpha_4 H_D^V S}{N} \geq \frac{\alpha_2 H_D^V}{N}$,

$\left(\frac{\alpha_3 L_D^V S}{N} + \frac{\alpha_4 H_D^V S}{N} + \frac{\beta_4 VS}{N}\right) \geq 0$.

Numerical Simulations for the Model

The graphical simulations of the co-menace model are presented using numerical the values for the variables and the parameters as presented in tables 1 below. In the simulation, we investigate the changes of the individuals' classes and assess the effect of increased rehabilitation of low drug-abusers on some of the classes and also, assess the effect of rehabilitation of violent persons on some of the classes.

Table 1: Parameters and Variables of the Model.

Symbols	Descriptions	Values and References
τ	Recruitment rate of the population	0.1891 Nigeria Watch (2017)
β_1	Conversion rate from susceptible to low drug abusing persons	0.0497 [15]
β_2	Conversion rate from susceptible to high drug abusing persons	0.0512 Assumed
π	Modification parameter for β_2	1.25 Mushayabasa & Tapedzesa (2015)
β_3	Conversion rate from susceptible to violent individuals	0.21 Sooknanan, *et al.* (2013)
β_4	Rate at which susceptible persons become violence victim	0.061 Assumed
α_1	Rate at which low drug-abusers become violent as well	0.31 Assumed
α_2	Rate at which high drug-abusers become violent as well	0.37 Assumed
α_3	Rate at which susceptible persons become violence victims for interacting with low drug abusing and violent individuals	0.45 Assumed
α_4	Rate at which susceptible persons become violence victims for interacting with high drug abusing and violent individuals	0.39 Assumed
ε_1	Progression rate from low to high drug abusing individual	0.16 Santoro, *et al* (2013)
ε_2	Progression rate from low to high drug abusing and violent individual as well	0.67 Santoro, *et al.* (2013)
ρ_1	Probability of high drug abusing persons going into rehabilitation	0.337 Assumed
ρ_2	probability of low drug abusing persons going into rehabilitation	0.2360 [15]
ρ_3	Probability of violent individuals going into rehabilitation	0.115 Sooknanan, *et al.* (2013)
θ	Rate at which rehabilitated persons become susceptible again	0.3 Mushayabasa & Tapedzesa (2015)
σ_1	Death rate caused by low drug-abuse	0.035 Mushayabasa & Tapedzesa (2015)
σ_2	Death rate induced by high drug-abuse	0.14 Mushayabasa & Tapedzesa (2015)
σ_3	Vices induced death rate of individuals in rehabilitation	0.4 Santoro, *et al*. (2013)
δ_1	Rate at which low drug abusing as well as violent persons die in course of the violence	0.0321 Santoro, *et al.* (2013)
δ_2	Rate at which high drug abusing as well as violent persons die in course of the violence	0.222 Santoro, *et al.* (2013)
δ_3	Rate at which violent individuals die in the course	0.83 Assumed
δ_4	Rate at which violence victims die for being violated	0.127 Assumed
μ	Rate of natural death in the entire population	0.0200 [15]
S_0	Susceptible persons at time t	11,700,000 Santoro, *et al.* (2013)
L_{D0}	Low drug abusing individuals at time t	900,000 Santoro, *et al.* (2013)
H_{D0}	High drug-abusers at time t	340,000 Santoro, *et al.* (2013)
L_{D0}^{V}	Low drug-abusers and violent at time t	350,000 Santoro, *et al.* (2013)
H_{D0}^{V}	High drug-abusers and violent at time t	800,000 Santoro, *et al.* (2013)
V_0	Violent individuals at time t	648,000 Assumed
V_{V0}	Violence victims at time t	130,000 Assumed
R_0	Rehabilitated individuals at time t	229,630 Santoro, *et al.* (2013)

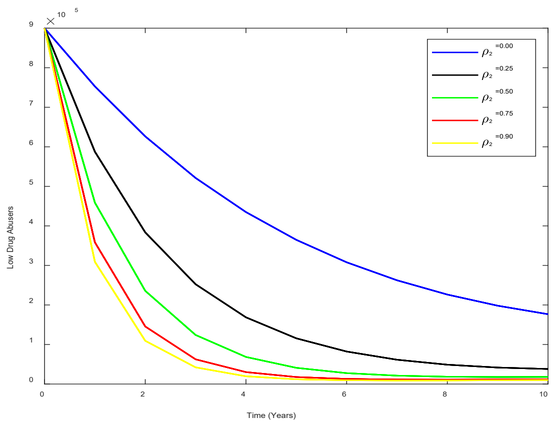

Figure 2: Impact of rehabilitation rates of low drug-abusers on their sub-population. The following parameter values were used: $\rho_2 = (0.00,\ 0.90)$.

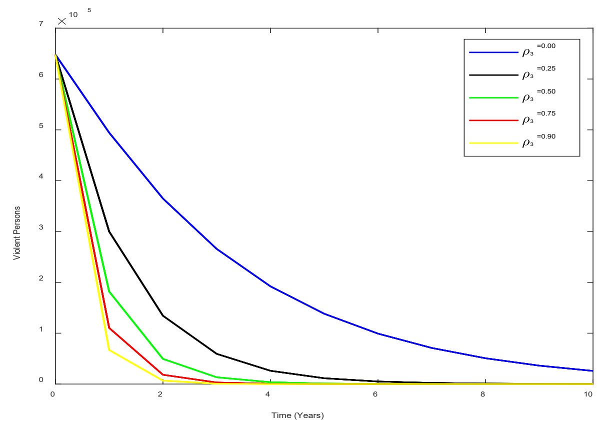

Figure 5: Impact of rehabilitation rates of violent persons' sub-population. The following parameter values were used $\rho_3 = (0.00,\ 0.90)$.

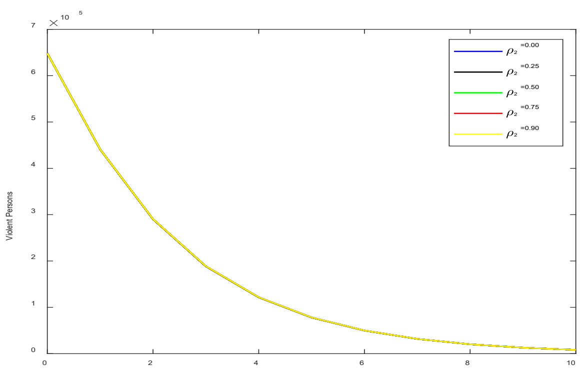

Figure 3: Impact of rehabilitation rates of low drug-abusers on violent persons' sub-population. The following parameter values were used: $\rho_2 = (0.00,\ 0.90)$.

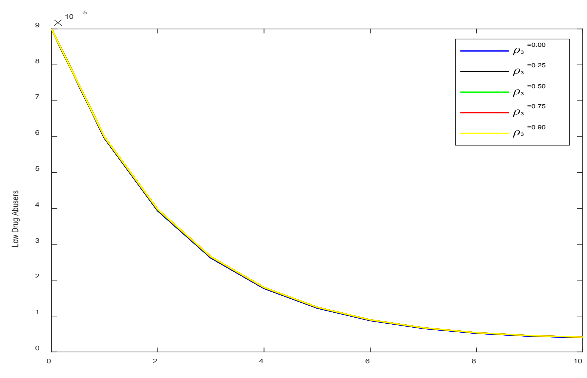

Figure 6: Impact of rehabilitation rates of violent persons on low drug-abusers' sub-population. The following parameter values were used $\rho_3 = (0.00,\ 0.90)$.

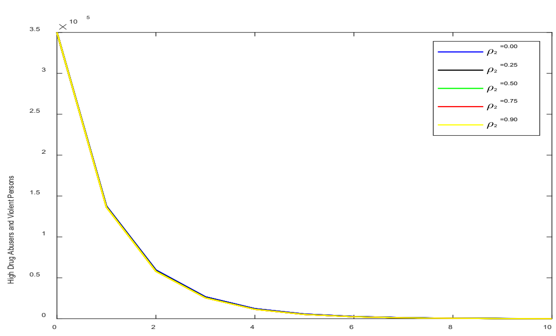

Figure 4: Impact of rehabilitation rates of low drug-abusers on high drug-abusers and violent persons' sub-population. The following parameter values were used: $\rho_2 = (0.00,\ 0.90)$.

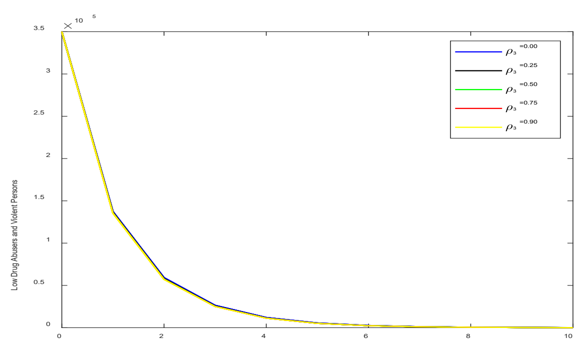

Figure 7: Impact of rehabilitation rates of violent persons on low drug-abusers and violents' sub-population. The following parameter values were used $\rho_3 = (0.00,\ 0.90)$.

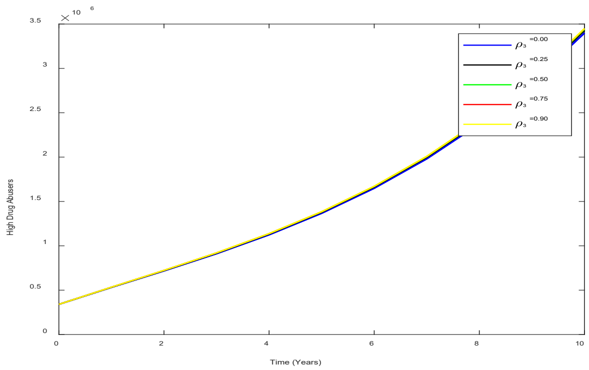

Figure 8: Impact of rehabilitation rates of violent persons on high drug-abusers' sub-population. The following parameter values were used $\rho_3 = (0.00,\ 0.90)$.

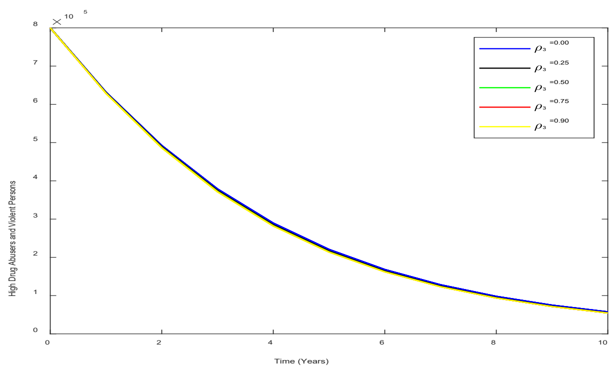

Figure 9: Impact of rehabilitation rates of violent persons on high drug-abusers and violents' sub-population. The following parameter values were used $\rho_3 = (0.00,\ 0.90)$.

DISCUSSION OF RESULTS

The drug-abuse-free equilibrium state for the drug-abuse sub-model is locally asymptotically stable; and similarly, it is globally asymptotically stable if the conditions stated are satisfied. The violence-free equilibrium state for the violence sub-system is locally asymptotically stable; and similarly, it is globally asymptotically stable if the conditions stated are satisfied. The co-menace-free equilibrium point for the co-menace model is locally asymptotically stable; and similarly, globally asymptotically stable if the stated conditions are satisfied.

For the numerical simulations, the followings are deduced: Figure 2 indicates that effect of rehabilitating low drug-abusers reduces their sub-population, in particular. Figure 3 indicates that controlling low drug-abuse only, has no significant effect on violence (because persons could be low drug-abusers but not violent). Figure 4 indicates that controlling low drug-abuse only, has no significant effect on high drug-abuse and violence (because persons could be low drug-abusers but not high drug-abusers and violent). Figure 5 indicates that increased rehabilitation of violent persons significantly makes their sub-population a decay function. Figure 6 reveals that more rehabilitation of violent persons has no significant effect on low drug-abusers' sub-population (for persons could be violent without being low drug-abusers). Figure 7 shows that more effort in controlling violence only, has no significant effect on low drug-abusers and violent persons. Figure 8 reveals that more rehabilitation of violent persons has no significant effect on high drug-abusers' sub-population (because persons could be violent without being high drug-abusers). Figure 9 shows that controlling violence only, has no significant effect on high drug-abusers and violent persons.

CONCLUSION

In this work, we mathematically modeled the dynamics of drug abuse and violence co-menace with the view to understand and control the menace in the population. The drug abuse, violence and drug abuse – violence co-menace free equilibrium points were obtained. Local and global stability analysis were carried out and the result shows that the equilibrium point of the system is locally and globally asymptotically stable if some conditions are met.

The result of the numerical computation carried out indicates that rehabilitating low drug-abusers and rehabilitation of violent persons reduces the population of low drug users and hard drug users. Hence, an effective way of controlling the drug abuse and violence co-menace in the population.

CONFLICT OF INTEREST

The authors declared that there is no conflict of interest.

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