Sine Diffusion Model for Nonlinear Dynamical Systems

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ABSTRACT

A dynamical system is a concept in mathematics where a fixed rule describes how a part in a geometrical space depends on time. Dynamical systems are mathematical objects used to model physical phenomena whose state changes over time. The realizations from these systems are usually corrupted by random noise. Hence, the importance of modelling such systems as a stochastic process. This study considered modelling the systems using stochastic differential equations which is a diffusion model. The study proposed a logarithmic mean reverting sine diffusion (LMRSD) process for modelling dynamics of nonlinear dynamical systems. Two empirical data were used for illustrations and the Euler scheme in R statistical package was adopted for the numerical estimations of the model parameters. The proposed model was compared with existing models and the study revealed that the proposed model performs well than the existing models with lowest values of the information criterion.

(Keywords: dynamical systems, stochastic differential equations, mean reversion, Euler scheme).

INTRODUCTION

A dynamical system is a concept in mathematics where a fixed rule describes how a part in a geometrical space depends on time. Dynamical systems are mathematical objects used to model physical phenomena whose state (or instantaneous description) changes over time.

These models are used in physics, mathematics, engineering, financial and economic forecasting, environmental modeling, medical diagnosis, industrial equipment diagnosis, and a host of other applications. Many problems in theoretical economics and finance are mathematically formalized as dynamical systems of difference and differential equations. The characteristics of dynamical systems are characteristics of mathematical models, and these can be linear, nonlinear, deterministic, stochastic, discrete, and continuous (Kalman, 1960).

Dynamical systems are very common and are considered to be stochastic processes by virtue of their own mechanism. By treating them as stochastic processes, meaningful results both in theory and applications may be obtained. During the last few decades, interest in the study of stochastic phenomena has increased dramatically and intense research activity in this area has been stimulated by the need to take into account random processes (Akintunde et al, 2015). The realizations from dynamical systems are usually corrupted by random noise. Hence, the importance of modelling such systems as a stochastic process.

The applications of dynamical systems become highly nontrivial when dynamics is nonlinear in the (Gaussian) parameter functions. This happens naturally for nonlinear systems which are driven by a Gaussian noise process, or when the nonlinearity is needed to provide necessary contrasts (e.g. positively) for the parameter functions (Akintunde et al, 2015).

Nonlinear systems have a wide range of behaviours which is much greater than that for linear systems. The behaviors of nonlinear dynamical system are in different forms and range from very simple periodic solutions to complicated "chaotic" behavior (Devaney, 1989). The systems are characterized by a lot of uncertainties which need to be well captured. The stochastic aspects of the models are used to capture the uncertainty about the environment in which the system is operating and the structure and parameters of the models of physical processes being studied (Shali, 2012).

A lot of studies and research has been carried out in modeling nonlinear dynamical systems. Such works include that of Akintunde et al, (2015). In their study, nonlinear dynamical system was modeled as a solution to an Ito stochastic differential equation. Other works include that of Mukhin, *et al*. (2006), Agwuegbo, *et al.* (2011), Mowery (1965), Neal (1968), Lainiotis (1971), and Hall, *et al*. (2012) just to mention a few. Recent research has seen a significant increase in interest in fitting nonlinear differential equation to data. Many systems of differential equations used to describe real world phenomena are developed from first-principles approaches by a combination of conservation laws and rough guesses. Such *a prior*i modelling has led to proposed models that mimic the qualitative behavior of observed systems, but have poor quantitative agreement with empirical measurements.

The problem of modeling dynamics of nonlinear systems has become an active field due to its potential applications especially in finance and engineering and is viewed as a realization from stochastic process of a nonlinear dynamical system. In the stochastic formulation, the dynamic behavior is modeled as a stochastic differential equation (Akintunde, 2015). This study was motivated by the work of Cai and Pan (2017) and therefore considered modelling the systems using stochastic differential equations which is a sine diffusion model.

MATERIALS AND METHODS

Let X_t be the realization from a dynamical system. A dynamical system can be specified by a state vector and a function. The state vector gives a numerical description of the underlying phenomenon of the system, while the function is a rule which gives how the system changes over time. The dynamical systems are of the form:

$$\dot{X} = f(X,t), \quad t \in \mathbb{R} \qquad (1)$$

for continuous time and

$$X(t+1) = f[X(t),t],$$

$$t \in \boldsymbol{Z} \qquad (2)$$

for discrete time. Where X is the dynamical variable and t, is the time parameter. If the right hand sides of Equations (1) and (2) are nonlinear functions, then we have nonlinear dynamical system.

When modelling a dynamic system, one is interested in the dynamics of the underlying phenomenon of the system as a function of time. The states of the system form a succession of random steps as they evolve with time. This implies that the system can be said to follow a Random walks which can be represented as:

$$X_{t+1} = X_t + a_{t+1} \qquad (3)$$

Where a_{t+1} represents a white-noise term with a mean of zero, constant variance and zero auto-correlation. X_t are independent from each other and can also be described to include a trend, μ. Then, (3) becomes a random walk model with trend μ which can be written as:

$$X_{t+1} = \mu + X_t + a_{t+1} \qquad (4)$$

The relative change of the process as the system evolves with time is therefore given as:

$$X_{t+1} - X_t = \mu + a_{t+1} \qquad (5)$$

The dynamical system is considered to be nonlinear of the form:

$$\dot{X} = f(X,t) \qquad (6)$$

Such that.

$$f(\bar{X}_t) = 0 \qquad (7)$$

This implies that the system has a fixed point, $\bar{X}_t$ and the system is expected to move towards achieving equilibrium position or stability on the long run. Then, a process of mean reversion can be introduced. The mathematical phenomena of mean reversion with a modification to the random walk assumption as illustrated by Blanco and Soronow (2001) can be used to extend (5) such that the system is modelled as:

$$X_{t+1} - X_t = k(\mu - X_t) + \sigma X_t a_{t+1} \qquad (8)$$

Where μ is the mean reversion level or long run equilibrium state of the system, k is the mean reversion rate, σ the volatility and a is the random shock within the system from t to t+1. The system sample path is said to drift towards the mean reversion level, at a speed determined by the mean reversion rate.

Modelling in discrete time as a mean-reverting model, one can defined the underlying stochastic property of the system as a stochastic differential equation which is an Ornstein-Uhlenbeck process (Agwuegbo. et.al., 2017). The mean-reverting Ornstein-Uhlenbeck process can be given as:

$$dX_t = k(\mu - X_t)dt + \sigma X_t dW_t, \quad t \geq 0 \tag{9}$$

Where α, $\mu \in R$, σ >0, and W_t is a standard Wiener process. The solution to (9) has mean $\mu \in R$ and is mean-reverting and the parameter σ represents the degree of volatility around the mean, which is caused by stochastic shocks and the process is said to be a mean reverting process that exhibits a pronounced tendency toward an equilibrium value with initial value, X_0 (Agwuegbo et.al., 2017). The components of (9), $k(\mu - X_t)dt$ is the drift and $\sigma X_t dW_t$ is the diffusion coefficient.

However, several behaviors are possible in nonlinear dynamic system. These include single stable steady state, multi-stability, cycle oscillations and some complex behavior such as chaos. In order to take care of the different behaviours and uncertainty inherent in the system, the drift and diffusion are assumed a general function of the system and time. Assuming a logarithmic drift process, we have a mean reverting logarithmic process (Cai and Pan, 2017) given as

$$d\ln X_t = k(\mu - \ln X_t)dt + \sigma X_t dW_t, \quad t \geq 0 \tag{10}$$

The present study proposed a modification of (10) as logarithmic mean reverting process written as

$$d\ln X_t = k(\mu - \ln X_t)dt + \sigma \ln X_t dW_t, \quad t \geq 0 \tag{11}$$

In addition to the modification, we also assume an introduction of a sine function of the diffusion coefficient of (11) to handle the cyclic behavior that can be exhibited by the nonlinear system. This leads to our proposed LMRSD model given as:

$$d\ln X_t = k(\mu - \ln X_t)dt + \sigma \sin(\ln X)_t dW_t, \quad t \geq 0 \tag{12}$$

Letting $k = 1$ and $Y_t = \ln X_t$, (12) becomes

$$dY_t = (\mu - Y_t)dt + \sigma \sin Y_t dW_t, \quad t \geq 0 \tag{13}$$

This gives a stochastic differential equation (SDE) which is a diffusion process.

Parameter Estimation for the SDE

The study considered estimation of the model parameters using the Euler-Maruyama estimation scheme/technique for one-dimensional SDE. Given step-size Δt, and setting initial value

$$y_0 = Y(0), \quad Y(0) \geq 0 \tag{14}$$

The Euler scheme produced the discretization $\Delta t \to 0$

$$Y_{t+\Delta t} - Y_t = (\mu - Y_t)\Delta t + (\sigma \sin Y_t)(W_{t+\Delta t} - W_t), \tag{15}$$

The increments $Y_{t+\Delta t} - Y_t$ are then independent Gaussian random variables with mean

$$E_y = (\mu - Y_t)\Delta t \tag{16}$$

and variance

$$V_y = [(\sigma \sin Y_t)(W_{t+\Delta t} - W_t)]^2 \Delta t \tag{17}$$

With (14) and Δt, we obtain approximations $y_n \approx Y(t_n)$, where $t_n = n\Delta t$ given as:

$$y_{n+1} = y_n(1-\Delta t) + \mu\Delta t + \sigma \sin y_n \Delta W_n \quad (18)$$

where

$$\Delta W_n = W_{n+1} - W_n \quad (19)$$

Given the values y_n and Δt, one can estimate the unknown parameters of the model in (13).

The study made use of the Sim.DiffProc package of R statistical package Guidoum and Boukhetala (2015). The package implements Pseudo-Maximum likelihood Estimation (PMLE) methods via the *fitsde* function. The interface and the output of the *fitsde* function are made as similar as possible to those of the standard MLE function in the stats4 package of the basic R system. The main arguments to *fitsde* consist of a data which is a univariate time series and initial values for optimization.

The proposed model was compared with the existing models which are based on combining many assumptions for the drift and diffusion components of stochastic differential equations. For the selection of the preferred model, the information criteria used in the study was the Akaike Information Criteria (AIC) (Akaike, 1974).

AIC is a measure of the relative quality of a statistical model for a given data set. It estimates the quality of a model, relative to the other competing models. It gives a relative estimate of the information lost when a given model is used to represent the process that generates the data. Therefore, the model with the least AIC value is preferred.

RESULTS AND DISCUSSIONS

The proposed LMRSL model was applied, and comparison was made with the existing MRL model. Two data sets were used in this study. The empirical data used are the Nigerian All Shares Index data (1985 to 2014) and the Exchange rate (Nigerian Naira to US Dollars) data (1980-2014). Figures 1 and 2 show the plot of the realizations from the Nigerian All Shares Index data and the Exchange rate (Nigerian Naira to US Dollars) data, respectively.

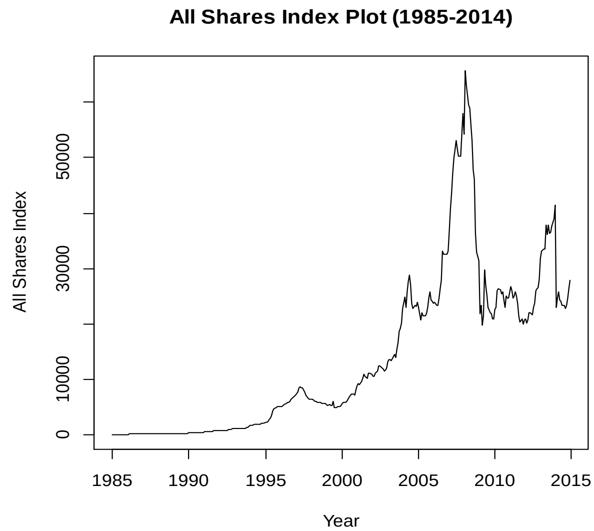

Figure 1: All Shares Index Plot.

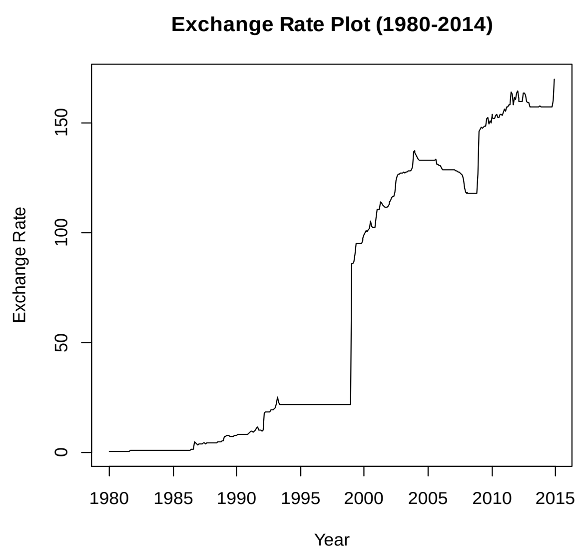

Figure 2: Exchange Rate Plot.

Figures 3 and 4 show the autocorrelation plots of Nigerian All Shares Index data and the Exchange rate (Nigerian Naira to US Dollars) data respectively while Figures 5 and 6 show the partial autocorrelation plots of Nigerian All Shares Index data and the Exchange rate (Nigerian Naira to US Dollars) data, respectively.

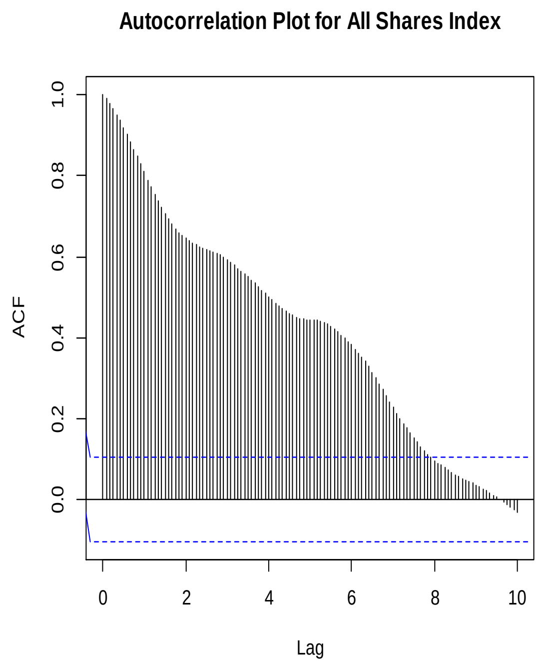

Figure 3: ACF for All Shares Index.

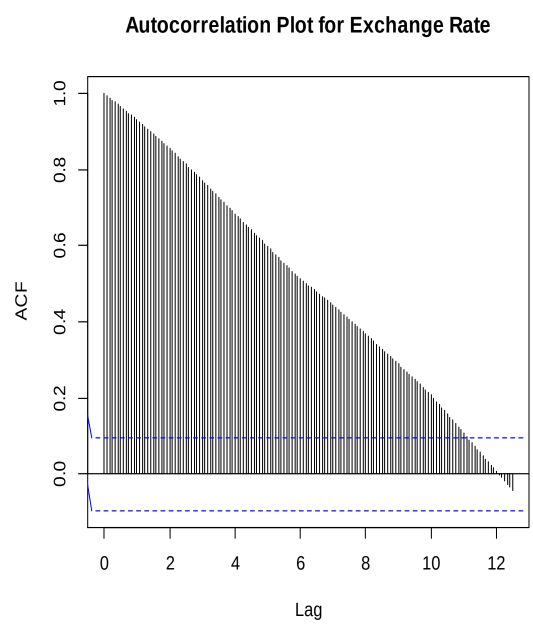

Figure 4: ACF for Exchange Rate

The plots (Figure 1 to 6) show that the underlying processes are nonstationary and high volatile with possible jumps.

Tables 1 and 2 gave a description of the performance of the proposed model, logarithmic mean reverting sine diffusion process (LMRSDP), in comparison with the other existing models. The existing SDE models used are mean reverting process (MRP), mean reverting logarithmic process (MRLP) and mean reverting square root process (MRSRP).

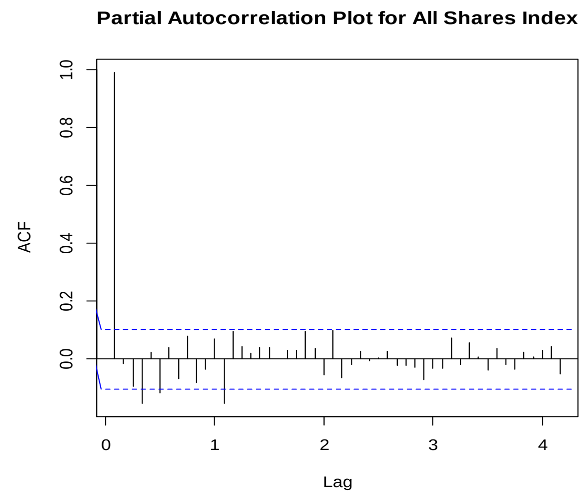

Figure 5: PACF for All Shares Index.

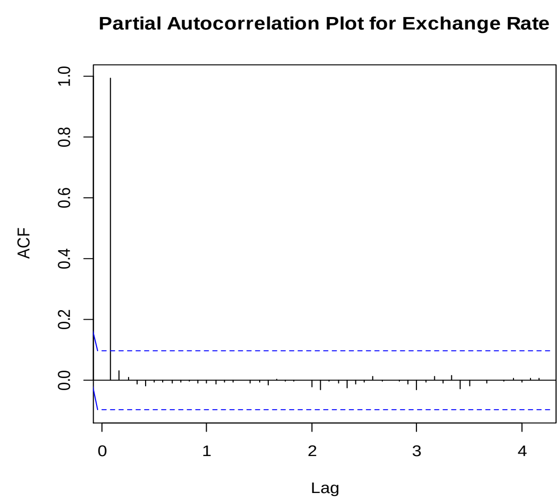

Figure 6: PACF for Exchange Rate

From Tables 1 and 2, MRSRP is seen to be equally good but not consistent in performance as it is selective while LMRSDP (the proposed model) is consistent in performance. The proposed model therefore can be considered to perform well than the other existing models for nonlinear systems.

Table 1: Performance of the Proposed Model in Comparison with the other Existing Models for the Exchange Rate Data.

Models	Theta1	Theta2	Log likelihood	AIC	Performance Position
MRP	0.7921952	0.9106252	-1834.238	3672.476	3rd
MRLP	3.455986	4.088002	-1184.511	2373.022	2nd
MRSRP	3.310521	8.002915	-2105.44	4214.879	4th
LMRSDP	0.2393686	104.2700176	-681.0288	1366.058	1st

Table 2: Performance of the Proposed Model in Comparison with the other Existing Models for the All Shares Index Data.

Models	Theta1	Theta2	Log likelihood	AIC	Performance Position
MRP	0.8898476	0.8898476	-3491.83	6987.66	4th
MRLP	85.77529	1822.56279	-3204.771	6413.541	3rd
MRSRP	635.58976	--50.46013	0	4	1st
LMRSDP	-373.9167	7173.7938	-1342.251	2688.502	2nd

CONCLUSION

The present study proposed a logarithmic mean reverting sine diffusion model, which is a stochastic differential equation, for modeling nonlinear dynamical systems. The proposed model consistently performs best among the models compared. Therefore, the proposed model is recommended for the model and analysis of nonlinear systems. The study also recommends the use of the fitsde codes of Sim.DiffProc package in R statistical package for easy parameter estimation of the said proposed model.

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