Mathematical Modeling of Algae Population Dynamics on the Surface of Water

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ABSTRACT

The paper presented an analytical solution of the exponential growth model of algae population dynamics on the water surface. The Computer Symbolic Algebraic Package, MAPLE is used to simulate the graphical profiles of the population with time while varying the parameters, such as diffusion and rate of change of algae density, governing the subsistence or extinction of the water organisms.

(Keywords: exponential growth, phytoplankton, diffusion, source term, sink term)

INTRODUCTION

Algae are a diverse group of aquatic organisms that have the ability to conduct photosynthesis. Certain algae are familiar to most people; for instance, seaweeds (such as kelp or phytoplankton), pond scum, or the algal blooms in lakes. However, there exists a vast and varied world of algae that are not only helpful to us, but are critical to our existence (Aparna, 2016).

Some species of algae are edible and are used as gelling agent in some food. Most algae are found in freshwater and marine environments; a few grow in terrestrial habitats. The algae are not a single, closely related taxonomic group but, instead, are a diverse assemblage of unicellular, colonial, and multicellular eukaryotic organisms.

Although algae can be autotrophic or heterotrophic, most are photo-autotrophs. They store carbon in a variety of forms, including starch, oils and various sugars (Microsoft Cooperation, 2003). The body of algae is called the thallus. Algae thalli range from small solitary cells to large, complex multicellular structures. Algae reproduce asexually and sexually.

There is substantial evidence for the health benefits of algal-derived food products, but there remain considerable challenges in quantifying these benefits, as well as possible adverse effects. First, there is a limited understanding of nutritional composition across algal species, geographical regions, and seasons, all of which can substantially affect their dietary value (Mark et al, 2017). Some other types contain harmful toxins and are hazardous to health. When found in drinking water, algae can make the process of filtration more complex and costlier.

Algae are needed in aquaria and lakes to create a balanced ecosystem but can constitute problems if its growth is not controlled.

A Model may be defined as a simplified or idealized descriptions or conception of a particular system, situation or process. It may be categorized according to the medium in which they are expressed.

Mathematical modeling is the process of creating a mathematical representation of some phenomenon in order to gain a better understanding of that phenomenon (Benyah, 2005). It has become an important scientific technique over the last three decades and is becoming more and more a powerful tool to solve problems arising from science, engineering, economics, industries and the society in general.
A differential equation is an equation containing one or more derivatives of the unknown function. Differential equation form very important mathematical tool used in developing models of physical and biological processes as well as formulating significant problems in the social sciences. If y is a dependent variable, x is an independent variable and the rate of growth or decay of y depends directly on its present state or value; then Ordinary Differential Equation in his simplest form is given as (Akinwande, 2006).

$$\frac{dy}{dx} = \varphi y \qquad y(x_0) = y_0 \tag{1}$$

Where φ is constant.

As the functional relation between the derivative of y and the variables *x* and *y.* equation (1) has the solution:

$$y(x) = y_0 \ell^{\varphi(x - x_0)} \tag{2}$$

If $\varphi > 0$ then *y* is steady growing while if $\varphi < 0$ it is decaying.

If *P(t)* is the population of community at time t, the Malthusians law which proposed the exponential growth model for population is of the form in Equation (1) above given:

$$\frac{dp(t)}{dt} = \delta p(t) \tag{3}$$

Where δ is the growth modulus.

We note the following on the proposed exponential model Equation (3):

Note that (a) if we define $\beta =$ Birth rate and $\mu =$ death rate then

$$\delta = \beta - \mu \tag{4}$$

Hence, the population decay exponentially then $\delta < 0$ and it grows exponentially when $\delta > 0$ (b) The model in Equation (3) does not apply where the population competes for space and resources in which case the growth modulus may depend on population. (Akinwande, 2006)

MODEL FORMULATION

We considered the environment inhabited by the algae to be closed in the sense that there is no migration into and out of the space it occupies. Since each member of the population reproduces at the same rate *(r).* The total growth rate is then:

$$\frac{dN}{dt} = rN \tag{5}$$

Where *N(t)* is the population level at time *t.*

Equation (5) suggests that the population apparently grows without bound as *t* increases. Let us assume that the plankton reproduces at a growth rate *(r)* proportional to its present density $\rho(x,t)$. We then consider the diffusion model in which v is the intensity of diffusion; the rate of change of plankton density with time is given as:

$$\frac{\partial \rho}{\partial t} = v \frac{\partial^2 \rho}{\partial x^2} + R(x,t) \tag{6}$$

R(x,t) is the source term. We should note at this point that the exponential growth model considered internal reproduction as a source term. Let,

$$R(x,t) = r\rho(x,t) \tag{7}$$

be the source term. If we put Equation (7) into Equation (6) we have:

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} + r\rho \tag{8}$$

Equation (8) is the model equation for the exponential population growth model

Method of Solution

In using separation of variable to solve Equation (8) we let:

$$\rho(x,t) = A(x)B(t) \tag{9}$$

$$\frac{\partial \rho}{\partial t} = A(x)\frac{dB}{dt} = A'(x)\dot{B}(t) \tag{10}$$

$$\frac{\partial \rho}{\partial x} = \frac{dA}{dx}B(t) = A'(x)B(t) \tag{11}$$

$$\frac{\partial^2 \rho}{\partial x^2} = \frac{d^2 A}{dx^2}B(t) = A''(x)B(t) \tag{12}$$

Substituting (9), (10) and (12) into (8) gives:

$$A(x)\dot{B} = \nu A''(x)B(t) + rA(x)B(t) \quad \textbf{(13)}$$

Dividing through by $\nu A''(x)B(t)$ gives:

$$\frac{1}{\nu}\frac{B(t)}{B(t)} = \frac{A''(x)}{A(x)} + \frac{r}{\nu} \quad \textbf{(14)}$$

$$\frac{1}{\nu}\frac{B(t)}{B(t)} - \frac{r}{\nu} = \frac{A''(x)}{A(x)} = \lambda \quad \textbf{(15)}$$

Since the two sides of Equation (15) contain two independent variables *x* and *t.* these can only be equal if they are both equal to an identical constant λ.

From Equation (15):

$$\frac{A''(x)}{A(x)} = \lambda \quad \textbf{(16)}$$

$$\Rightarrow A''(x) - \lambda A(x) = 0 \quad \textbf{(17)}$$

And

$$\frac{1}{\nu}\frac{\dot{B}(t)}{B(t)} - \frac{r}{\nu} = \lambda \quad \textbf{(18)}$$

Multiplying Equation (18) through by $\nu B(t)$:

$$\dot{B}(t) - rB(t) = \lambda \nu B(t) \quad \textbf{(19)}$$

$$\dot{B}(t) - (\lambda \nu + r)B(t) = 0 \quad \textbf{(20)}$$

To solve equation (20) using separation of variables:

$$\int \frac{\dot{B}(t)}{B(t)} dt = \int (\lambda \nu + r)\, dt \quad \textbf{(21)}$$

$$InB(t) = (\lambda \nu + r)t; \quad \textbf{(22)}$$

$$B(t) = K_0 \ell^{(\lambda \nu + r)t} \quad \textbf{(23)}$$

For Equation (23) to be equal to the growth rate, λ has to be non-positive therefore, let $\lambda = -\alpha^2$. Equation (23) then becomes:

$$B(t) = K_0 \ell^{(r - \alpha^2 \lambda)t} \quad \textbf{(24)}$$

To solve Equation (17) taking $\lambda = -\alpha^2$ and let,

$$A(x) = m^0$$

$$\Rightarrow A''(x) + \alpha^2 A(x) = 0 \quad \textbf{(25)}$$

$$m^2 + \alpha^2 = 0 \quad \textbf{(26)}$$

$$m^2 = -\alpha^2 \quad \textbf{(27)}$$

$$m = \pm i\alpha \quad \textbf{(28)}$$

$$m = c_1 \cos \alpha x + c_2 \sin \alpha x \quad \textbf{(29)}$$

$$A(x) = c_1 \cos \alpha x + c_2 \sin \alpha x \quad \textbf{(30)}$$

Recall that $\rho(x,t) = A(x)B(t)$

$$\rho(x,t) = \ell^{(r - \alpha^2 \lambda)t}(c_1 \cos \alpha x + c_2 \sin \alpha x) \quad \textbf{(31)}$$

We model the effect of patch boundaries by stipulating the boundary conditions:

$$\rho(0,t) = \rho(L,t) = 0$$

$$\rho(0,t) = \ell^{(r - \alpha^2 \lambda)t} c_1 = 0 \Rightarrow c_1 = 0 \quad \textbf{(32)}$$

$$\rho(L,t) = \ell^{(r - \alpha^2 \lambda)t}(c_2 \sin \alpha L) = 0 \quad \textbf{(33)}$$

From Equation (33) if,

$c_2 = 0 \Rightarrow \rho(x, \mathrm{t}) = 0$,

which is not acceptable as a solution since the patch density cannot be zero. We then consider:

$$\sin \alpha L = 0 \quad \textbf{(34)}$$

$$\alpha L = \pi \quad \textbf{(35)}$$

$$\alpha = \frac{\pi}{L} \qquad \textbf{(36)}$$

Subsisting $c_1 = 0$; $c_2 = 1$ into Equation (33) we have:

$$\rho(x,t) = \ell^{(r-\alpha^2 v)t}(\sin\alpha x) \qquad \textbf{(37)}$$

Equation (37) gives the solution ρ which represents the density of the algae. To obtain the total population *N(t)* at time *t,* we integrate Equation (37) with respect to *x* using the boundary conditions to obtain the limit of integration thus:

$$N(t) = \int_0^L \rho(x,t)\,dx = Exp[(\mathrm{r}-\alpha^2\,\mathrm{v})\,\mathrm{t}]\int_0^L .Sin\alpha x dx$$

(38)

$$N(\mathrm{t}) = Exp[(\mathrm{r}-\alpha^2\,\mathrm{v})\,\mathrm{t}]\int_0^L Sin\alpha x dx \qquad \textbf{(39)}$$

$$N(\mathrm{t}) = Exp[(r-\alpha^2 v)t]\left[\frac{-\cos\alpha\,\mathrm{x}}{\alpha}\right]_0^L \qquad \textbf{(40)}$$

$$N(t) = \frac{Exp\left[r-\left(\frac{\pi}{L}\right)^2 v\right]t}{\alpha} \qquad \textbf{(41)}$$

Putting Equation (36) into Equation (41) we have:

$$N(t) = \frac{L}{\pi}Exp\left[r-\left(\frac{\pi}{L}\right)^2 v\right]t\left(1-\cos\frac{\pi}{L}L\right) \textbf{(42)}$$

$$N(t) = \frac{L}{\pi}Exp\left[r-\left(\frac{\pi}{L}\right)^2 v\right]t\left(1-\cos\pi\right) \textbf{(43)}$$

$$N(t) = \frac{2L}{\pi}Exp\left[r-\left(\frac{\pi}{L}\right)^2 v\right]t \qquad \textbf{(44)}$$

$$N(\mathrm{t}) = \frac{2L}{\pi}Exp\left(r-\left(\frac{\pi}{L}\right)^2 v\right)t \qquad \textbf{(45)}$$

If $\left(r-\left(\frac{\pi}{L}\right)^2 v<0\right)$, $N(t)\to\infty$ as $t\to\infty$ **(46)**

$$\Rightarrow r < \left(\frac{\pi}{L}\right)^2 v$$

(47)

$$\left(\frac{\pi}{L}\right)^2 > \frac{r}{v}$$

(48)

$$\frac{\pi}{L} > \sqrt{\frac{r}{v}}$$

(49)

$$\frac{L}{\pi} > \sqrt{\frac{v}{r}}$$

(50)

$$L < \pi\sqrt{\frac{v}{r}} = L_{\min}$$

(51)

The population of the Algae goes into extinction if $L < L_{\min}$. For population sustenance $L > L_{\min}$.

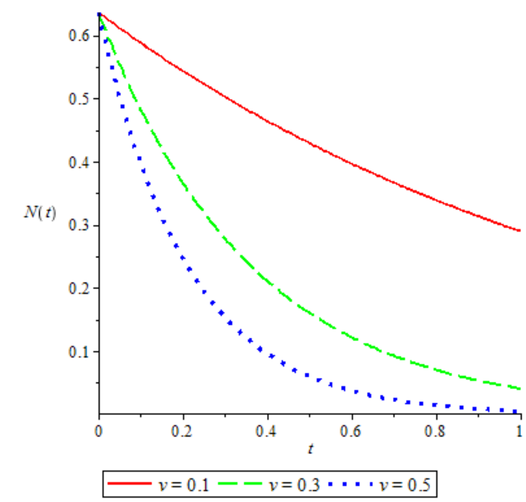

Figure 1: Variation of Algae Population $N(t)$ with Intensity of Diffusion v**.**

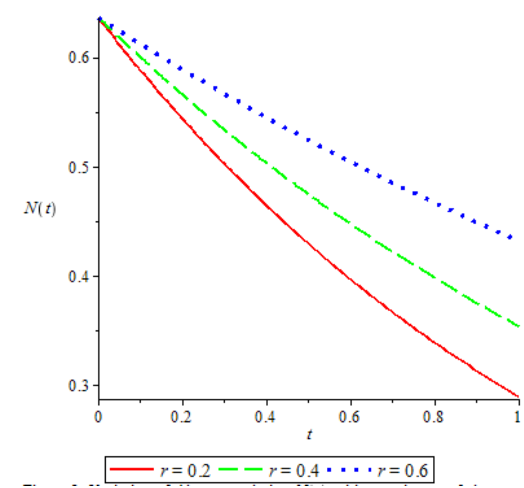

Figure 2: Variation of Algae Population $N(t)$ with Rate of Change of Algae Density r.

DISCUSSION

Figure 1 depicts the graph of algae population N(t) against time t for various values of the intensity of diffusion v. It is observed that algae population decreases with time and decreases as the intensity of diffusion increases.

Figure 2 displays graph of algae population N(t) against time t for various values of rate of change of algae density r. It is observed that algae population decreases with time but increases as the rate of change of algae density increases.

CONCLUSION

The owners of fish ponds and swimming pools will need the results of this work to reduce the growth of algae in various water bodies. The intensity of diffusion should be increased and the plankton density reduces.

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