On Some Notes on the Engel Expansion of Ratios of Sequences Obtained from the Sum of Digits of Squared Positive Integers

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ABSTRACT

Objective: To introduce a new novel approach to the understanding of Engel expansions of ratios of number sequences.

Methodology: Let C be the ratios of consecutive elements of sequence obtained from the sum of digits of squared positive integers and D be the ratios of consecutive elements of sequence that is the complement of C but the domain is the positive integer. Engel series expansions and Pierce expansion were obtained for C while only Engel series expansions were obtained for D.

Findings: The distance between the respective Engel series and their means for the two sequences are neither unique nor uniformly distributed. The finite Engel series elements of the mean may not be the member of the series. The research will help in examining the amount of variance or divergence or how different rational numbers varies in a given sequence.

(Keywords: Engle expansion, Pierce expansion, digit sum, ratios)

INTRODUCTION

The Engel expansion, named after Fredrich Engel [1] of a positive real number x is the unique non-decreasing sequence of positive integers $\{a_1, a_2, a_3, a_4, ...\}$ such that;

$$x = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \frac{1}{a_1 a_2 a_3 a_4} + ... \quad \textbf{(1)}$$

Also the notion of Engel expansion was extended to non-zero real number [2] and in p-adic and other non-archimedean fields [3], and [4] modified the use to continued fraction.

The sequence used in this paper is the result of the digit sum of squared positive integers [5]. Similar works done on the sum and iterative digit sums of selected integer sequences can be found in the cases of cubed positive integers [6], Sophie Germain and Safe primes [7], Palindromic, Repdigit and Repunit numbers [8], Fibonacci numbers [9] and so on.

The sequence generated by the sum of digit of squared positive integers [5] is given as:

$$1, 4, 7, 9, 10, 13, 16, 18, 19, ... \quad \textbf{(A)}$$

And the sequence obtained from the complement of (A) is:

$$2, 3, 5, 6, 8, 11, 12, 14, ... \quad \textbf{(B)}$$

The ratio of two consecutive elements of sequence A is given as:

$$\frac{4}{1}, \frac{7}{4}, \frac{9}{7}, \frac{10}{9}, \frac{13}{10}, \frac{16}{13}, ... \quad \textbf{(C)}$$

The ratio of two consecutive elements of sequence B is given as:

$$\frac{3}{2}, \frac{5}{3}, \frac{6}{5}, \frac{8}{6}, \frac{11}{8}, \frac{12}{11}, ... \quad \textbf{(D)}$$

The sequence C converges to almost one with a mean of 1.11200275 which can be represented in Engel series as:

$$(1, \; 9, \; 125) \quad \textbf{(E)}$$

The sequence D converges to almost one with a mean of 1.101494025 which can be represented in Engel series as:

$$(1,\ 10,\ 67) \quad \textbf{(F)}$$

The aim of this paper is to investigate the behavior of the Engel series expansions of sequences C and D. The rationale behind the calculation of the mean between the Engel series and the mean is as results of [10-12]. The distance is calculated using the following procedures;

MATERIALS AND METHODS

Definition 1

Let the Engel series of a ratio of sequence C be (a_1, a_2, a_3) and the Engel series of the mean of the sequence C be $(1,\ 9,\ 125)$. The distance can be calculated thus using;

$$d_1 = \sqrt{(a_1 - 1)^2 + (a_2 - 9)^2 + (a_3 - 125)^2} \quad \textbf{(2)}$$

Definition 2

Let the Engel series of a ratio of sequence D be (a_1, a_2, a_3) and the Engel series of the mean of the sequence D be $(1,\ 10,\ 67)$. The distance can be calculated thus using;

$$d_2 = \sqrt{(a_1 - 1)^2 + (a_2 - 10)^2 + (a_3 - 67)^2} \quad \textbf{(3)}$$

The details are in the result section of the paper. The research approach is similar to the findings of [13], [14], [15], but this paper is restricted to a ratio sequence and distance between the overall mean.

RESULTS AND DISCUSSION

The Engel series expansion of sequences C and D are obtained and the distances between the respective Engel series and the corresponding mean E and F.

Sequence Generated by the Sum of Digits of Squared Integers

The result of the Engel series expansions of sequence C is summarized in Table 1.

Table 1: The Engel Series Expansion of Elements of Sequence C.

C	Ratio	Engel Expansions	Distance from the average
4			
7	1.75	1, 1, -4	129.247824
9	1.285714286	1, 4, 7	118.1058847
10	1.111111111	1, 9	125
13	1.3	1, 3	125.1439172
16	1.230769231	1, 4, -13	138.09055
18	1.125	1, 8	125.0039999
19	1.055555556	1, 18	125.3235812
22	1.157894737	1, 6, -19	144.0312466
25	1.136363636	1, 7, -22	147.0136048
27	1.08	1, 13, 25	100.079968
28	1.037037037	1, 27	126.2893503
31	1.107142857	1, 9, -28	153
34	1.096774194	1, 10, -31	156.0032051
36	1.058823529	1, 17	125.2557384
37	1.027777778	1, 36	127.8827588
40	1.081081081	1, 12, -37	162.0277754
43	1.075	1, 13, -40	165.0484777
45	1.046511628	1, 22, 43	83.02409289
46	1.022222222	1, 45	130.0807442
49	1.065217391	1, 15, -46	171.1052308
52	1.06122449	1, 16, 1	124.1974235
54	1.038461538	1, 26	126.1507035
55	1.018518519	1, 54	132.8533026
58	1.054545455	1, 18, -55	180.2248596
61	1.051724138	1, 19, -58	183.2730204
63	1.032786885	1, 31, 61	67.67569726
64	1.015873016	1, 63	136.1653407
67	1.046875	1, 21, 64	62.16912417
70	1.044776119	1, 22, -67	192.4396009
72	1.028571429	1, 35	127.6753696
73	1.013888889	1, 72	139.9785698
76	1.04109589	1, 24, -73	198.5673689
79	1.039473684	1, 25, -76	201.6358103
81	1.025316456	1, 39, -79	206.1940833
82	1.012345679	1, 81	144.2532495
85	1.036585366	1, 27, -82	207.7811349
88	1.035294118	1, 28, -85	210.857772
90	1.022727273	1, 44	129.8075499
91	1.011111111	1, 90	148.9496559
94	1.032967033	1, 30, -91	217.0184324
97	1.031914894	1, 31, -94	220.102249
99	1.020618557	1, 48, -97	225.3996451
100	1.01010101	1, 99	154.029218
103	1.03	1, 33, -100	226.2763797
106	1.029126214	1, 34, -103	229.3665189
108	1.018867925	1, 53	132.5179233
109	1.009259259	1, 108	159.4553229
112	1.027522936	1, 36, -109	235.5525419
115	1.026785714	1, 37, -112	238.6482768
117	1.017391304	1, 58, 115	50.009999
118	1.008547009	1, 117	165.1938256
121	1.025423729	1, 39, -118	244.8448488
124	1.024793388	1, 40, -121	247.9455585

126	1.016129032	1, 62	135.7718675
127	1.007936508	1, 126	171.2133172
130	1.023622047	1, 42, -127	254.1515296
133	1.023076923	1, 43, -130	257.2566812
135	1.015037594	1, 66, -133	264.221498
136	1.007407407	1, 35	127.6753696
139	1.022058824	1, 45, -136	263.471061
142	1.021582734	1, 46, -139	266.5801943
144	1.014084507	1, 71	139.5313585
145	1.006944444	1, 144	183.9836949
148	1.020689655	1, 48, -145	272.8021261
151	1.02027027	1, 49, -148	275.9148419
153	1.013245033	1, 76, 151	71.86793444
154	1.006535948	1, 153	190.6856051
157	1.019480519	1, 51, -154	282.1435805
160	1.01910828	1, 52, -157	285.259531
162	1.0125	1, 80	143.756739
163	1.00617284	1, 162	197.5702407

The result of the Engel series expansions of sequence D is summarized in Table 2.

All the elements of the finite Engel series expansions are all the elements of the sequence D. The small values of the distance between the Engel series and the mean values are small which indicates the close proximity between the mean and the individual series.

There is a unique Engel expansion for every rational numbers. This has been revealed by the sequence. The mean is not uniformly distributed. The randomness of generation of the Engel series can also be a motivation for the generation of pseudorandom numbers.

CONCLUSION

This research has shown that not only that every positive rational number has a unique finite Engel expansion [3], the complement of a sequence can differ by Pierce expansion. The distance between the respective Engel series expansions and the average Engel series is not unique. The reason why sequence D has only Engel expansion is subject to further research. The closer the numerator and the denominator of rational numbers are to each other, the smaller their distance from the mean. The finite Engel series value of the means may not be the elements of the parent sequence.

Table 2: The Engel Series Expansion of Sequence D.

D	Ratio	Engel Expansions	Distance from the average
3	1.5	1, 2	67.47592163
5	1.666666667	1, 2, 3	64.49806199
6	1.2	1, 5	67.18630813
8	1.333333333	1, 3,	67.36467917
11	1.375	1, 3, 8	59.41380311
12	1.090909091	1, 11,	67.00746227
14	1.166666667	1, 6	67.11929678
15	1.071428571	1, 14,	67.11929678
17	1.133333333	1, 8, 15	52.03844733
20	1.176470588	1, 6, 17	50.15974482
21	1.05	1, 20	67.74215822
23	1.095238095	1, 11, 21	46.01086828
24	1.043478261	1, 23	68.24954212
26	1.083333333	1, 12	67.0298441
29	1.115384615	1, 9, 26	41.01219331
30	1.034482759	1, 29,	69.64194139
32	1.066666667	1, 15	67.18630813
33	1.03125	1, 32	70.51950085
35	1.060606061	1, 17, 33	34.71310992
38	1.085714286	1, 12, 35	32.06243908
39	1.026315789	1, 38	72.61542536
41	1.051282051	1, 20, 39	29.73213749
42	1.024390244	1, 41	73.8241153
44	1.047619048	1, 21	67.89698079
47	1.068181818	1, 15, 44	23.53720459
48	1.021276596	1, 47	76.53757247
50	1.041666667	1, 24,	68.44705983
51	1.02	1, 50	78.0320447
53	1.039215686	1, 26, 51	22.627417
56	1.056603774	1, 18, 53	16.1245155
57	1.017857143	1, 56	81.27115109
59	1.035087719	1, 29, 57	21.47091055
60	1.016949153	1, 59	83.00602388
62	1.033333333	1, 30	69.92138443
65	1.048387097	1, 21, 62	12.08304597
66	1.015384615	1, 65	86.68333173
68	1.03030303	1, 33	70.83784299
69	1.014705882	1, 68	88.6171541
71	1.028985507	1, 35, 69	25.07987241
74	1.042253521	1, 24, 71	14.56021978
75	1.013513514	1, 74	92.65527508

The results of this research can be helpful in determining the differences or variance of closely related rational numbers which can appear as a sequence in this case. When applied to this research, we can conclude that no individual elements of sequence C are closed to the mean value and as such none of the elements can be taken as a representative of the sequence. Also the randomness of the results of the Engel expression can be applied in the generation of pseudo-random numbers.

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