A Fifth Stage Runge-Kutta Method for the Solution of Ordinary Differential Equations.

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ABSTRACT

In this paper, we developed a Hybrid Block method at $k = 2$ through the collocation approach and the block discrete schemes obtained were reconstructed to an effective Fifth Stage Implicit Runge-Kutta method for the solution of initial value problem of first order differential equations. The new approach shows superiority over its equivalent linear multi-step method (LMM) with numerical experiments tested with the method.

(Keywords: LMM, linear multi-step method, fifth stage, Runge-Kutta, first order initial value problem)

INTRODUCTION

In many real–life situations, the differential equations which model the problems are too complicated to solve exactly, therefore one of two approaches is taken to approximate the solutions. The first is to simplify the differential equation to one that can be solved exactly and then use the solution of the simplified equation to approximate the solution to the original equation. The other approach, the one we shall investigate in this paper, involves finding Runge-Kutta type method of equivalent to similar linear multistep of its kind for the solution of:

$$y'(x) = f(x, y) \quad , \quad y(0) = y_0$$

DEFINITIONS: RUNGE-KUTTA METHODS

The Runge-Kutta integration method for (1) is given by:

$$y_{n+1} = y_n + h\left(\sum_{j=1}^{s} b_j k_j\right)$$

Where,

$$k_i = f\left(x + c_i, y + \textstyle\sum_{j=1}^{s} a_{ij} k_j\right) \ i = 1,2, \ldots s \qquad \textbf{(1)}$$

(see Butcher 1996)

This method can be expressed in Table 1 (Butcher's Table).

$A = \left(a_{ij}\right) i, j = 1,2, \ldots . s$ is in S x S Matrix.

$$b^T = (b_1, b_2, b_3, \ldots \ldots b_s)$$

$$C_i = \sum_{j=1}^{s-1} a_{ij} \qquad \textbf{(2)}$$

Table 1: Butcher's Table.

C			A	
C_1	a_{11}	a_{12}	.	a_{1s}
C_2	a_{21}	a_{22}	.	a_{2s}
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
C_s	a_{s1}	a_{s2}	.	a_{ss}
b^T	b_1	b_2	.	b_s

$$= \begin{array}{c|c} C & A \\ \hline & b^T \end{array}$$

Linear Multi-Step Method (LMM)

Consider the initial value problem:

$$y' = f(x,y) \quad y(x_0) = y_0 \tag{3}$$

The linear multistep method is to find a polynomial of the form:

$$U(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j}(x) + \sum_{j=0}^{s-1} h\psi_j(x) f(c_j, u(c_j)) \tag{4}$$

where t denotes the number of interpolation points $x_j, j = 0, \ldots t-1$ and $s = m$ denote the distinct Collocation points $c_j, j = 0, \ldots s-1, \alpha_j(x), \psi_j(x)$ are to be determined by Lagrange interpolation conditions at suitable points, h is the step size vector, h can also be variable or constants.

The functions $\alpha_j(x)$ and $\psi_j(x)$ in (4) can be represented by polynomials of the form:

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_j x^i \quad j\epsilon \{0,1 \ldots . t-1\} \tag{5}$$

$$h\psi_j(x) = h \sum_{i=0}^{t+m-1} (i+1)\psi_j x^{i-1}, j\epsilon \{0,1 \ldots . s-1\} \tag{6}$$

Substituting (6) and (5) into (4) we have:

$$U(x) = \sum_{j=0}^{t+m-1} \left\{ \sum_{j=0}^{t-1} \alpha_j y_{n+j} + \sum_{j=0}^{m-1} h\psi_j f_{n+j} \right\} x^i = \sum_{i=0}^{t+m-1} a_i x^i \tag{7}$$

where,

$$a_i = \left(\sum_{j=0}^{t-1} \alpha_j y_{n+j} + \sum_{j=0}^{m-1} h\psi_j f_{n+j} \right) \quad a_i \in \Re^j, j = (0,1 \ldots t+m-1),$$

$$y \in C^m(a,b)$$

Definition Root Conditions

A LMM of (4) is said to satisfy the root conditions if all of the roots of the first characteristics polynomial have modulus less than or equal to unity and those of modulus unity are simple. The method (4) is said to be zero-stable if it satisfies the root condition (Lambert 1973).

Definition Explicit and Implicit Schemes

A LMM (4) is explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$, for an explicit method yields the current value y_{n+k} directly in terms of $y_{n+j}, f_{n+j}\ j = 0,1,\ldots k-1$, which, at this stage of the computation have already been calculated. Also an implicit method, however will call for the solution at each stage of the computation of the equation $y_{n+k} = h\beta_k f(x_{n+k}, y_{n+k}) + g$, where g is a known function of the previous calculated values of $y_{n+j}, f_{n+j}\ j = 0,1,..k-1$ (Butcher 1987).

Definition Zero-Stable

A linear multi-step method (4) is said to be Zero-stable if no root of the first characteristic polynomial $\rho(x) = \sum_{j=0}^{k} \alpha_j r^j$ has modulus greater than one and if every root with modulus one is simple.

DERIVATION OF THE METHOD

Given a differential equation:

$$y' = f(x, y),\ y(x_o) = y_o \qquad \textbf{(8)}$$

we consider a polynomial of the form:

$$U(x) = \sum_{j=0}^{m+t-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{m+t-1} \beta_j(x) f_{n+j}(\bar{x}, y(\bar{x}_j)) \qquad \textbf{(9)}$$

where t denotes the number of interpolation points, $x_{n+j},\ j = 0,1 \ldots t-1$, m the distinct collocation points $\bar{x}_j,\ j = 0,1 \ldots m-1$, y and f are smooth real functions. We can represent $\alpha_j(x)\ and\ \beta_j(x)$ by polynomial of the form:

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,} x^i, i = j = 0,1 \ldots t-1 \qquad \textbf{(10)}$$

$$\beta_j(x) = h \sum_{i=0}^{m+t-1} i\beta_{j,} x^{i-1}, j = 0,1 \ldots m-1 \qquad \textbf{(11)}$$

with constant coefficients α_j and β_j to be determined. Now we assume a power series solution of the form as a basis solution for (8). Also the second derivative of (12) gives:

$$y(x) = \sum_{j=0}^{m+t-1} \alpha_j x^j \qquad \textbf{(12)}$$

$$y'(x) = \sum_{j=0}^{m+t-1} j a_j x^{j-1} \tag{13}$$

Interpolating (12) at $x = x_n$ and collocating (13) at $x = x_{n+j}$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$. Specifically for this method $t = 1$ and $m = 5$, the degree of the polynomial is $m + t - 1$ and we have the following system of non linear equations:

$$a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 = y_n$$

$$a_1 + 2a_2 x_n + 3a_3 x_n^2 + 4a_4 x_n^3 + 5a_5 x_n^4 = f_n$$

$$a_1 + 2a_2 x_{n+\frac{1}{2}} + 3a_3 x_{n+\frac{1}{2}}^2 + 4a_4 x_{n+\frac{1}{2}}^3 + 5a_5 x_{n+\frac{1}{2}}^4 = f_{n+\frac{1}{2}}$$

$$a_1 + 2a_2 x_{n+1} + 3a_3 x_{n+1}^2 + 4a_4 x_{n+1}^3 + 5a_5 x_{n+1}^4 = f_{n+1}$$

$$a_1 + 2a_2 x_{n+\frac{3}{2}} + 3a_3 x_{n+\frac{3}{2}}^2 + 4a_4 x_{n+\frac{3}{2}}^3 + 5a_5 x_{n+\frac{3}{2}}^4 = f_{n+\frac{3}{2}}$$

$$a_1 + 2a_2 x_{n+2} + 3a_3 x_{n+2}^2 + 4a_4 x_{n+2}^3 + 5a_5 x_{n+2}^4 = f_{n+2} \tag{14}$$

By arranging (14) in Matrix equation form we have:

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+1/2} & 3x_{n+1/2}^2 & 4x_{n+1/2}^3 & 5x_{n+1/2}^4 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 1 & 2x_{n+3/2} & 3x_{n+3/2}^2 & 4x_{n+3/2}^3 & 5x_{n+3/2}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix} \tag{15}$$

The continuous formula for (15) will be of the form:

$$y(x) = \alpha_0 y_n + h\left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2}\right] \tag{16}$$

Using Maple 11 Mathematical software to evaluate the values of :

$a_j, (j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2)$ in (2.08) and substituted in (2.05) to obtain our Continuous formula for this method as:

$$y(x) = y_n + \left[(x - x_n) - \frac{25}{12h}(x - x_n)^2 + \frac{35}{18h^2}(x - x_n)^3 - \frac{5}{6h^3}(x - x_n)^4 + \frac{2}{15h^4}(x - x_n)^5\right] f_n$$

$$+\left[\frac{4}{h}(x-x_n)^2-\frac{52}{9h^2}(x-x_n)^3+\frac{3}{h^3}(x-x_n)^4-\frac{8}{15h^4}(x-x_n)^5\right]f_{n+\frac{1}{2}}$$

$$+\left[-\frac{3}{h}(x-x_n)^2+\frac{19}{3h^2}(x-x_n)^3-\frac{4}{h^3}(x-x_n)^4+\frac{4}{5h^4}(x-x_n)^5\right]f_{n+1}$$

$$+\left[\frac{4}{3h}(x-x_n)^2-\frac{28}{9h^2}(x-x_n)^3+\frac{7}{3h^3}(x-x_n)^4-\frac{8}{15h^4}(x-x_n)^5\right]f_{n+\frac{3}{2}}$$

$$+\left[-\frac{3}{12h}(x-x_n)^2+\frac{11}{18h^2}(x-x_n)^3-\frac{1}{2h^3}(x-x_n)^4+\frac{2}{15h^4}(x-x_n)^5\right]f_{n+2} \quad \textbf{(17)}$$

Evaluating (17) at $x = x_{n+j}, j = \frac{1}{2}, 1, \frac{3}{2}\ and\ 2$ to obtain the following discrete schemes as our Block method at $k = 2$:

$$y_{n+\frac{1}{2}} = y_n + \frac{251}{1440}hf_n + \frac{323}{720}hf_{n+\frac{1}{2}} - \frac{11}{60}hf_{n+1} + \frac{53}{720}hf_{n+\frac{3}{2}} - \frac{19}{1440}hf_{n+2}$$

$$y_{n+1} = y_n + \frac{29}{180}hf_n + \frac{31}{45}hf_{n+\frac{1}{2}} + \frac{2}{15}hf_{n+1} + \frac{1}{45}hf_{n+\frac{3}{2}} - \frac{1}{180}hf_{n+2}$$

$$y_{n+\frac{3}{2}} = y_n + \frac{27}{160}hf_n + \frac{51}{80}hf_{n+\frac{1}{2}} + \frac{9}{20}hf_{n+1} + \frac{21}{80}hf_{n+\frac{3}{2}} - \frac{3}{160}hf_{n+2}$$

$$y_{n+2} = y_n + \frac{7}{45}hf_n + \frac{32}{45}hf_{n+\frac{1}{2}} + \frac{4}{15}hf_{n+1} + \frac{32}{45}hf_{n+\frac{3}{2}} + \frac{7}{45}hf_{n+2} \quad \textbf{(18)}$$

The method (18) is of Order $[5,5,5,6]^T$ with error constant of $\left[\frac{3}{10240}, \frac{1}{5760}, \frac{3}{10240}, -\frac{1}{15120}\right]^T$.

Also (18) is arranged in Matrix form we have:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-\frac{3}{2}}\\y_{n-1}\\y_{n-\frac{1}{2}}\\y_n\end{bmatrix}+\begin{bmatrix}\frac{323}{720}&\frac{-11}{60}&\frac{53}{720}&\frac{-19}{1440}\\\frac{31}{45}&\frac{2}{15}&\frac{1}{45}&\frac{-1}{180}\\\frac{51}{80}&\frac{9}{20}&\frac{21}{80}&\frac{-3}{160}\\\frac{32}{45}&\frac{4}{45}&\frac{32}{45}&\frac{7}{45}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 & \frac{251}{1440} \\ 0 & 0 & 0 & \frac{29}{180} \\ 0 & 0 & 0 & \frac{27}{160} \\ 0 & 0 & 0 & \frac{7}{45} \end{bmatrix} \begin{bmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} \qquad \textbf{(19)}$$

when (19) is arranged in Butcher array, we have:

C	**A**				
$\frac{1}{4}$	$\frac{251}{2880}$	$\frac{323}{1440}$	$\frac{-11}{120}$	$\frac{53}{1440}$	$\frac{-19}{2880}$
$\frac{1}{2}$	$\frac{29}{360}$	$\frac{31}{90}$	$\frac{1}{15}$	$\frac{1}{90}$	$\frac{-1}{360}$
$\frac{3}{4}$	$\frac{27}{320}$	$\frac{51}{160}$	$\frac{9}{40}$	$\frac{21}{160}$	$\frac{-3}{320}$
1	$\frac{7}{90}$	$\frac{16}{45}$	$\frac{2}{15}$	$\frac{16}{45}$	$\frac{7}{90}$
$\boldsymbol{b} = 1$	$\frac{7}{90}$	$\frac{16}{45}$	$\frac{2}{15}$	$\frac{16}{45}$	$\frac{7}{90}$

(20)

The above table satisfies Runge-Kutta conditions for solution of first order ODEs since:

$$(i) \quad \sum_{j=1}^{s} a_{ij} = c_i$$

$$(ii) \quad \sum_{j=1}^{s} b_j = 1$$

The method (18) is formally given as Runge-Kutta type method as:

$$y_{n+1} = y_n + h\left(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5\right)$$

$$k_1 = f\left(x_n, y_n\right)$$

$$k_2 = f\left(x_n + \frac{1}{4}h, \quad y_n + h(\frac{251}{2880}k_1 + \frac{323}{1440}k_2 - \frac{11}{120}k_3 + \frac{53}{1440}k_4 - \frac{19}{2880}k_5)\right)$$

$$k_3 = f\left(x_n + \frac{1}{2}h, \quad y_n + h(\frac{29}{360}k_1 + \frac{31}{90}k_2 + \frac{1}{15}k_3 + \frac{1}{90}k_4 - \frac{1}{360}k_5 \right)$$

$$k_4 = f\left(x_n + \frac{3}{4}h, \quad y_n + h(\frac{27}{320}k_1 + \frac{51}{160}k_2 + \frac{9}{40}k_3 + \frac{21}{160}k_4 - \frac{3}{320}k_5 \right)$$

$$k_5 = f\left(x_n + h, \quad y_n + h(\frac{7}{90}k_1 + \frac{16}{45}k_2 + \frac{2}{15}k_3 + \frac{16}{45}k_4 + \frac{7}{90}k_5) \right)$$

(21)

STABILITY ANALYSIS OF THE METHOD

Thus obtain the normalized form of (19), the first characteristics polynomial of the normalized matrix will be expressed as:

$$\rho(R) = det\left[RA^{(0)} - A^{(1)}\right] = det\left[R\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} - \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}\right] = 0$$

$$det\left[R\begin{pmatrix}R & 0 & 0 & -1\\ 0 & R & 0 & -1\\ 0 & 0 & R & -1\\ 0 & 0 & 0 & R-1\end{pmatrix}\right] = R^3(R-1) = 0$$

which implies that $R_1 = R_2 = R_3 = 0$ and $R_4 = 1$. From definition (4) and Block method (19) is zero stable and also consistent as its Order $P = [5,5,5,6]^T > 1$, thus it is convergent.

IMPLEMENTATION STRATEGIES

The method (21) is applied on the given problems at $n = 0,1,2\ 3,\dots$ gives the following set of solutions at $y_1, y_2, y_2, \dots, y_{10}$, sequentially.

Example 1

$$y' = -30y, \qquad y(0) = \frac{1}{3} \qquad h = 0.1 \qquad 0 \le x \le 1.0$$

Exact Solution: $y(x) = \frac{1}{3}e^{-30x}$

Example 2

$$y' - 2y = e^{-x}, \qquad y(0) = \frac{3}{4} \qquad h = 0.1 \qquad 0 \le x \le 1.0$$

Exact Solution: $y(x) = \left(-\frac{1}{3}e^{-3x} + \frac{13}{12}\right)e^{2x}$

Example 3

$y' = 8(x - y) + 1$, $y(0) = 2$, $h = 0.01$

Exact solution: $y(x) = x + 2e^{-8x}$

Example 4

$y' = 1 - y^2$, $y(0) = 0$, $h = 0.1$

Exact solution: $y(x) = \frac{e^{2x}-1}{e^{2x}+1}$ or $\tanh(x)$

Example 5: (Logistic Population Model)

A logistic law has useful applications to human populations and animal populations. The mathematical model is defined as:

$$y(t) = Ay(t) - B\ y^2(t), \quad y(t_0) = g_0$$

A, B are positive constant. If B = 0, it gives the Malthus population model. If $0 < y(0) < \frac{A}{B}$, the population is monotonic increasing to a large population, to limit $\frac{A}{B}$, if, y(0) > $\frac{A}{B}$, it decrease to limit $\frac{A}{B}$.

Now we consider a logistic model of the form: $y' - Ay = -By^2$, $y(t_0) = y_0$, A, B positive constant, with $A = 5, B = 3,\ t_0 = 0, y_0 = 2$. Thus the model reduces to: $y' = 5y - 3y^2$, $y(o) = 2$

Table 2: Approximate Errors of Example 1.

MESH VALUES	YAKUBU ET AL (2004) K = 2	KWAMI (2011) K= 2	PRESENT METHOD K = 2
0.05		5.8738 X 10-3	4.36647 X 10-6
0.10	4.2376 X 10-3	3.1783 X 10-3	2.41783 X 10-6
0.15		1.8895 X 10-2	3.22127 X 10-7
0.20	8.2507 X 10-2	3.8721 X 10-2	2.33717 X 10-8
0.25		2.1560 X 10-3	5.60830 X 10-9
0.30	1.6949 X 10-3	7.4088 X 10-4	3.66663 X 10-9
0.35		8.8456 X 10-4	1.35706 X 10-9
0.40	6.9423 X 10-3	1.5619 X 10-3	4.23056 X 10-10
0.45		1.0577 X 10-4	1.21232 X 10-10
0.50	1.4457 X 10-4	3.0621 X 10-5	5.82400 X 10-12
0.55		3.5322 X 10-5	2.63465 X 10-12
0.60	5.7869 X 10-4	6.1849 X 10-5	8.85947 X 10-13
0.65		4.2361 X 10-6	2.64196 X 10-13
0.70	1.2056 X 10-5	1.2228 X 10-6	7.37922 X 10-14
0.75		1.3977 X 10-6	1.977727 X 10-14
0.80	4.8225 X 10-5	2.4462 X 10-6	5.151960 X 10-15
0.85		1.6757 X 10-7	1.314469 X 10-15
0.90	1.0046×10^{-6}	4.8371×10^{-8}	3.300963×10^{-16}
0.95		5.5282×10^{-8}	8.186580×10^{-17}
1.0	4.0187×10^{-6}	9.6743×10^{-8}	2.009904×10^{-17}

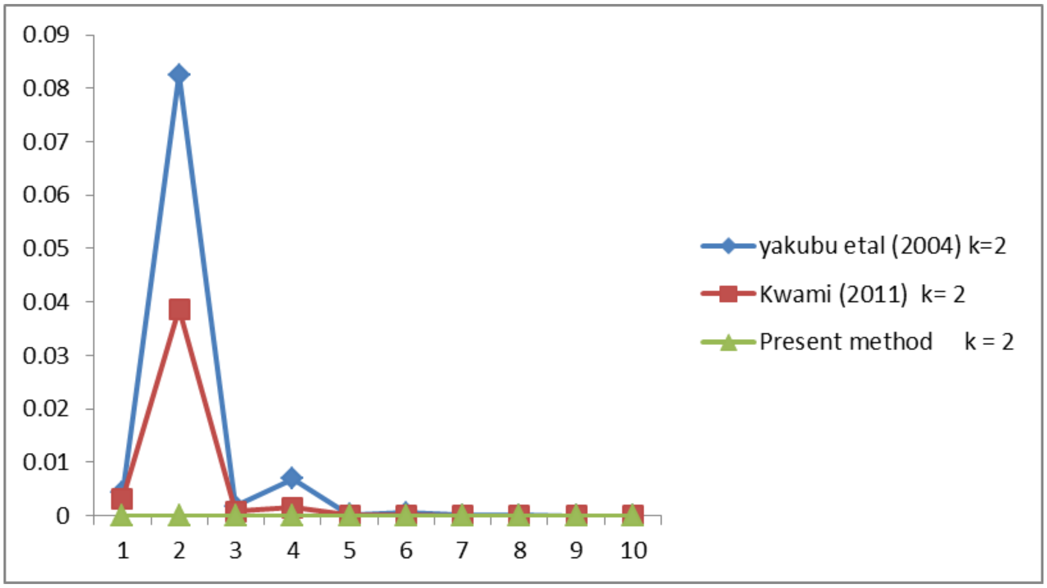

Figure 1: Error graph of Example 1.

Table 3: Approximate Errors of Example 2.

MESH VALUES	BADMUS AND MSHELIA(2011) AT K = 2	PRESENT METHOD K = 2
0.1	8.9965 E – 06	1.0 E – 09
0.2	1.6612 E – 06	1.0 E – 09
0.3	4.1717 E – 06	2.0 E – 09
0.4	2.1166 E – 06	3.0 E – 09
0.5	1.5667 E – 06	4.0 E – 09
0.6	2.0008 E – 06	3.0 E – 09
0.7	2.393 E – 06	3.0 E – 09
0.8	1.6569 E – 06	7.0 E – 09
0.9	2.0072 E – 06	1.0 E – 08
1.0	1.2623 E - 06	5.0 E – 09

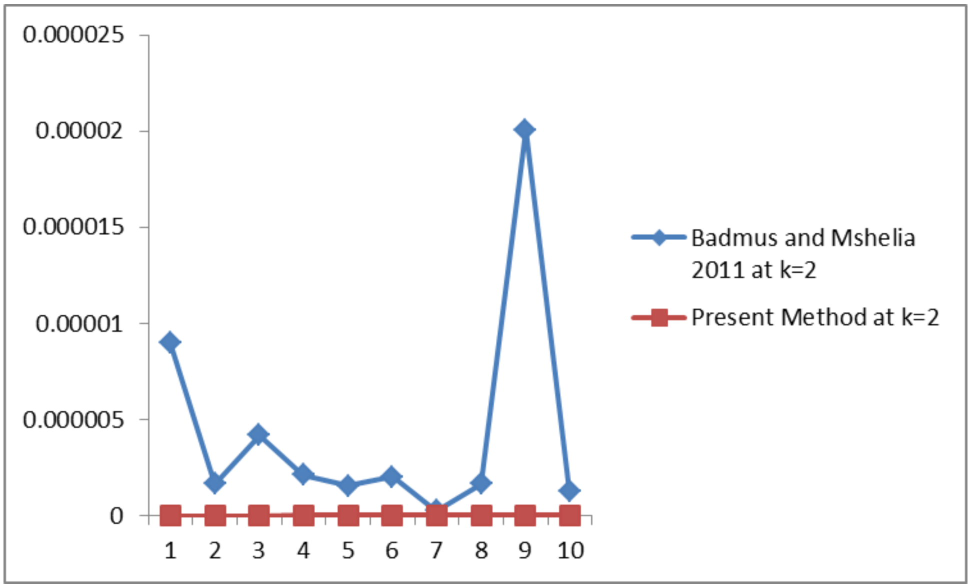

Figure 2: Error Graph of Example 2.

Table 4: Approximate Errors of Problem 3 with h=0.01.

MESH VALUES	ODEJIDE AND ADENIRAN (2012) AT K=5	BADMUS E TAL (2013) AT K=4	PRESENT METHOD AT K=2
0.1	8.07132 E (-14)	2.4 E (-14)	5.9 E (-13)
0.2	3.86469 E(-13)	3.1 E (-14)	5.29 E (-13)
0.3	1.44384 E(-13)	2.0 E (-14)	3.55 E (-13)
0.4	7.30527 E (-14)	2.4 E (-14)	2.14 E (-13)
0.5	3.863588 E (-14)	2.0 E (-14)	1.2 E (-13)
0.6	7.54952 E (-14)	3.0 E (-14)	6.6 E (-14)
0.7	2.34257 E(-14)	2.5 E (-14)	3.6 E (-14)
0.8	2.70894 E(-14)	4.0 E (-14)	1.9 E (-14)
0.9	4.57412 E (-14)	3.2 E (-14)	1.0 E (-14)
1.0	3.9523 E (-14)	4.0 E (-14)	0.00

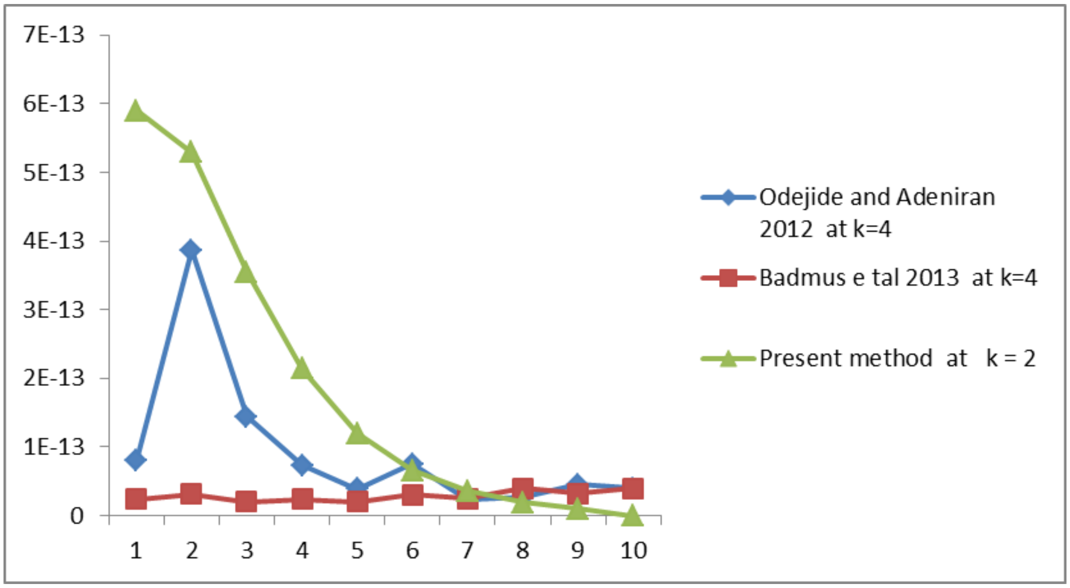

Figure 3: Error Graph of Example 3.

Table 5: Approximate Errors of Example 4.

MESH VALUES	ODEKUNLE et al 2012 at k=4	BADMUS et al 2015 at k=4	PRESENT METHOD K = 2
0.1	3.1386×10^{-7}	1.23125×10^{-9}	1.33919×10^{-11}
0.2	1.3364×10^{-7}	1.23433×10^{-9}	2.7014×10^{-11}
0.3	3.1819×10^{-7}	1.28142×10^{-9}	3.8094×10^{-11}
0.4	9.8972×10^{-9}	9.01000×10^{-12}	4.5675×10^{-11}
0.5	6.9521×10^{-7}	3.3949×10^{-9}	4.8518×10^{-11}
0.6	4.0794×10^{-7}	3.5428×10^{-9}	4.6234×10^{-11}
0.7	7.7319×10^{-7}	3.8244×10^{-9}	3.9483×10^{-11}
0.8	7.0821×10^{-7}	3.8275×10^{-10}	2.98112×10^{-11}
0.9	2.8682×10^{-6}	1.3275×10^{-8}	1.9051×10^{-11}
1.0	2.0664×10^{-6}	1.4427×10^{-8}	8.834×10^{-12}

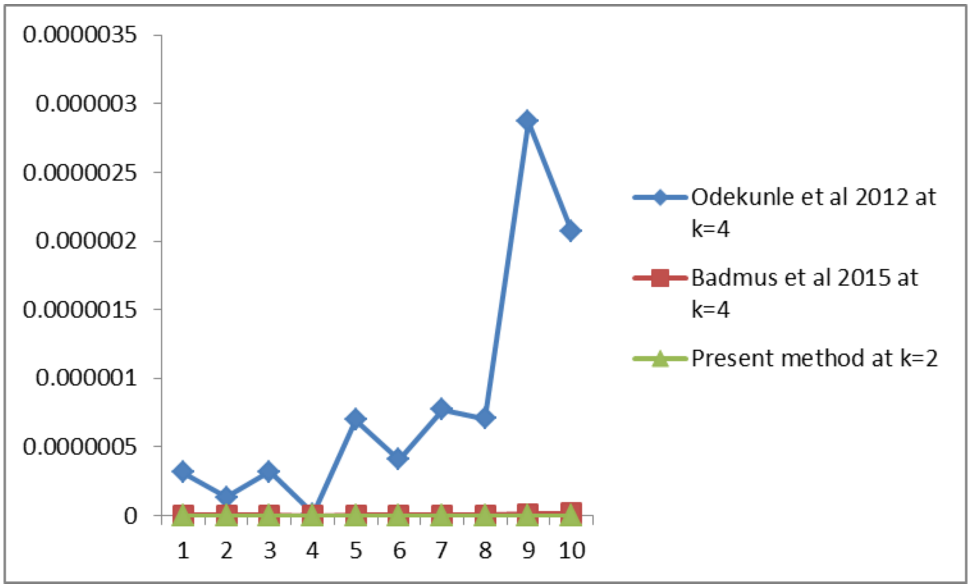

Figure 4: Error Graph of Example 4.

Table 6: Approximate Errors of Example 5.

MESH VALUES	LIE AND NORSETT (1989)	PRESENT METHOD AT K=2
0.2	1.20 X 10^{-5}	6.5171949 X 10^{-6}
0.4	9.26 X 10^{-7}	2.72406171 X 10^{-6}
0.6	4.84 X 10^{-7}	1.05876278 X 10^{-6}
0.8	2.14 X 10^{-7}	4.0687465 X 10^{-7}
1.0	9.05 X 10^{-8}	1.5582515 X 10^{-7}

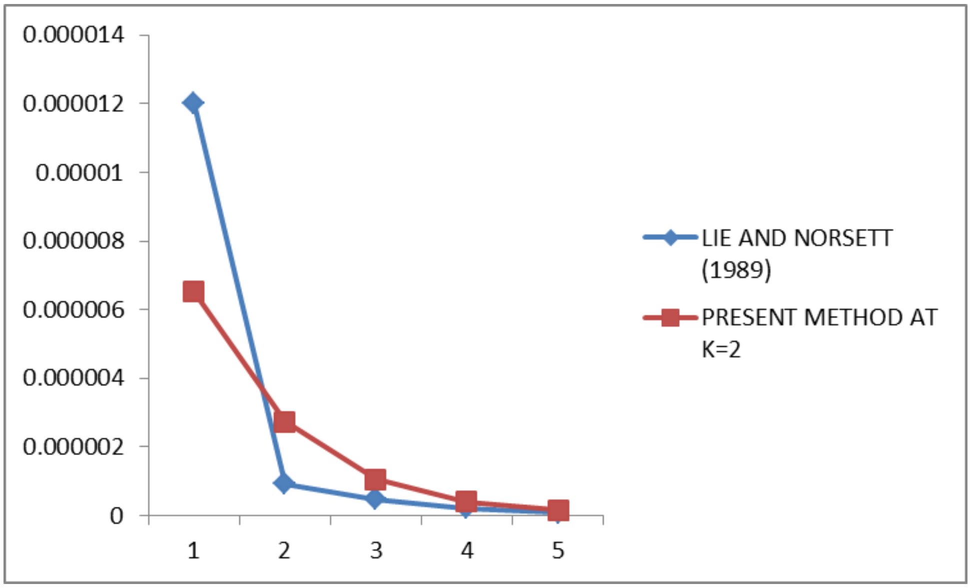

Figure 5: Error Graph of Example 5.

DISCUSSION OF RESULTS

Based on the numerical Experiments 1 and 2 solved, we observed that the new fifth stage Runge-Kutta method developed at K=2 performed excellently well with Yakubu et al. (2004) and Kwami (2011) (see Tables 1 and 2).

In Problem 3, our new method which is of lower order and lower step length of K=2 compared favourably with Badmus et al (2013), Odejide and Adeniran (2012) which are of uniform order 9 with higher step length of K=4 and 5, respectively. Also in Problems 4 and 5, our new method converges better than the existing methods (see Tables 4, 5 and Figures 5, 6).

CONCLUSION

We have derived a 5-stage fifth order implicit Runge-Kutta method for the solution of first order differential equations. The schemes are efficient, stable and implementation cost is minimal**.** We want to conclude that the new fifth stage Runge-Kutta method performed excellently well when compared with the existing methods. Also when implemented with a scheme of both higher order and step length, our new method still compete closely with their exact solutions.

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