An Orthogonal Based Self-Starting Numerical Integrator for Third Order IVPs in ODEs.

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ABSTRACT

This work focuses on construction of continuous approximation schemes for the numerical solution of third order initial value problems in ordinary differential equations. A set of orthogonal polynomials valid in interval [-1, 1] with respect to weight function $w(x)=1+\frac{x}{2}$ was constructed and exploited as basis function and, we derived from the continuous scheme an implicit hybrid block method through collocation at some selected points. Analysis of the proposed method shows that it is zero-stable, consistent and hence convergent. On comparison, the method performed favorably well in terms of computational time, function evaluation per step, cost of implementation and accuracy.

(Keywords: approximation schemes, numerical solutions, third order, ordinary differential equations, ODE)

INTRODUCTION

Mathematical modelling which leads to formulation of different equations is an important tool in solving Ordinary Differential Equations (ODEs) problems resulting from different fields such as the physical sciences, engineering technology, management, economics, and medicine.

In the physical sciences, the mathematical formulation of the study of rate of decomposition of radioactive substances, the problem of determining the motion of planets, satellites, rockets, and projectiles, and the conduction of heat through a medium do give rise to ODEs.

This work considers approximate methods for the solution of the general third order Initial Value Problems of the form:

$$y'''=f(x,y,y',y''),\ y^{k}(x_n)=y_n^{k},\ k=0,1,2 \qquad \textbf{(1)}$$

where x_n is the initial point, y_n is the solution at x_n, f is continuous within the interval of integration.

Problems involving Equation (1) were extensively discussed in Lambert (1973), Fatunla (1988), Awoyemi (1992) in the past by first reducing them to system of first order differential equations and then solve the resulting equations by any of the existing numerical methods. The disadvantage of which leads to greater computational cost.

However, direct method for solving (1) which were more efficient than the method of reduction to system of first order ordinary differential equations have been studied by many scholars including: Adeyefa et al (2014), Anake (2012), Adesanya et al (2012), Awoyemi et al (2011), Jator (2007) to mention a few.

The direct methods are self-starting methods which are formulated in terms of linear multistep methods called block methods. The block method provides the traditional advantage of one-step methods e.g., Runge-Kutta methods, of being self-starting and permitting easy change of step length (Lambert, 1973). Another important feature of the block approach is that all the discrete schemes are of uniform order and are obtained from a single continuous formula in contrast to the non-self starting predictor-corrector approach.

This self-starting method was used by Anake (2013) to derive a class of one-step hybrid methods for the numerical solution of second order ordinary differential equation with power series as the basis function. Lately, Adeyefa (2014) adopted this same approach but employed Chebyshev Polynomial to develop a set of algorithms. The numerical solutions obtained by these researchers are desirable as their methods at many points recovered the exact solutions. In what follows, we shall adopt block method approach to formulate a third order numerical scheme by developing orthogonal polynomials which are employed as basis function to derive a block method that provides direct solution to (1)

CONSTRUCTION OF ORTHOGONAL BASIS FUNCTION

We set out here to derive orthogonal polynomials using the weight function $w(x) = 1 + \frac{x}{2}$.

The procedure demands choosing the orthogonal polynomial $q_r(x)$ defined as:

$$q_r(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \qquad (2)$$

Where $C_r' s$ are the orthogonal coefficients and $q_r(x)$ satisfies the inner product:

$$< q_m(x), q_n(x) > = \int_a^b w(x) q_m(x) q_n(x) dx = 0, \quad m \neq n \qquad (3)$$

For the purpose of constructing the basis function, we use additional property:

that $q_r(1) = 1$ (4)

For r = 0 in (2),
$q_0(x) = C_0^{(0)}$

From (4),

$$q_0(1) = C_0^{(0)} = 1$$

Hence,

$$q_0(x) = 1$$

For r = 1 in (2),

$$q_1(x) = C_0^{(1)} + C_1^{(1)} x \qquad (5)$$

By definition (4), (5) gives:

$$C_0^{(1)} + C_1^{(1)} = 1 \qquad (6)$$

and

$$< q_0, q_1 > = \int_0^1 \left(1 + \frac{x}{2}\right) q_0(x) q_1(x) dx \qquad (7)$$

which implies:

$$\frac{5}{4} C_0^{(1)} + \frac{2}{3} C_1^{(1)} = 0 \qquad (8)$$

Solving (6) and (8) and substituting the outcomes into (5), we have:

$$q_1(x) = \frac{1}{7}(15x - 8) \qquad (9)$$

When r = 2 in (2),

$$q_2(x) = C_0^{(2)} + C_1^{(2)} x + C_2^{(2)} x^2 \qquad (10)$$

By definition (4), (10) gives:

$$C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1 \qquad (11)$$

and

$$< q_0, q_2 > = \int_0 \left(1 + \frac{x}{2}\right) q_0(x) q_2(x) dx = 0 \qquad (12)$$

which implies:

$$\frac{5}{4}C_0^{(2)} + \frac{2}{3}C_1^{(2)} + \frac{11}{24}C_2^{(2)} = 0$$

also,

$$< q_1, q_2 > = \int_0^1 \left(1 + \frac{x}{2}\right) q_1(x) q_2(x) dx \quad (14)$$

which gives:

$$\frac{37}{168}C_1^{(2)} + \frac{19}{84}C_2^{(2)} = 0$$

Solving (11), (13), (15) and substituting the resulting values into (10), we have:

$$q_2(x) = \frac{1}{57}(370x^2 - 380x + 67) \qquad (16)$$

When n = 3 in (2),

$$q_3(x) = C_0^{(3)} + C_1^{(3)}x + C_2^{(3)}x^2 + C_3^{(3)}x^3 \qquad (17)$$

By definition (4), (17) gives:

$$C_0^{(3)} + C_1^{(3)} + C_2^{(3)} + C_3^{(3)} = 1 \qquad (18)$$

$$< q_0, q_3 > = \int_0^1 \left(1 + \frac{x}{2}\right) q_0(x) q_3(x) dx = 0 \quad (19)$$

which implies:

(13)

$$\frac{5}{4}C_0^{(3)} + \frac{2}{3}C_1^{(3)} + \frac{11}{24}C_2^{(3)} + \frac{7}{20}C_3^{(3)} = 0$$

$$< q_1, q_3 > = \int_0^1 \left(1 + \frac{x}{2}\right) q_1(x) q_3(x) dx = 0 \quad (12)$$

This leads to:

$$\frac{37}{168}C_1^{(3)} + \frac{19}{84}C_2^{(3)} + \frac{29}{140}C_3^{(3)} = 0 \qquad (22)$$

(15)

$$< q_2, q_3 > = \int_0^1 \left(1 + \frac{x}{2}\right) q_2(x) q_3(x) dx = 0$$

(23)

Solving (18),(20) and (22) and substituting the resulting values into (17), we obtain:

$$q_3(x) = \frac{1}{491}(10675x^3 - 16290x^2 + 6690x - 584)$$

In the same vein, $q_n(x), n \geq 4$ are developed. The next three polynomials which are used in this work are listed hereunder.

$$q_4(x) = \frac{1}{4361}(332766x^4 - 674072x^3 + 440874x^2 - 100428x + 5221)$$

$$q_5(x) = \frac{1}{7899}(2173710x^5 - 5489736x^4 + 4942812x^3 - 1884904x^2 + 275513x - 9496)$$

$$q_6(x) = \frac{1}{72509}(73254324x^6 - 221626152x^5 + 254436138x^4 - 137426374x^3 + 34913052x^2 - 3565896x + 87419)$$

FORMULATION OF THE PROPOSED METHOD (PM)

The procedure to derive the continuous hybrid methods shall be considered in this section. Here, we set out by approximating the analytical solution of problem (i) with a polynomial of the form:

$$y(x)=\sum_{n=0}^{r+s-1}a_n q_n(x) \qquad (25)$$

where $q_n(x)$ is the orthogonal polynomials derived, $a_n's$ are the real unknown parameters to be determined, r is the number of collocation points, s is the number of interpolation points. We shall now employ the set of polynomials to formulate a continuous scheme through which numerical solutions of initial value problems in ordinary differential equations are obtained.

We consider here equation (25) to obtain the solution of (1) in the sub-interval [x_n, x_{n+p}] of [a, b] taking our basis function to be an orthogonal function where $x=\frac{2X-2x_n-ph}{ph}$ and p varies as the method to be derived.

Here, we are formulating a two-step method (i.e., p = 2 and, r and s are points of collocation and interpolation, respectively). The procedure involves interpolating (25) at $x=x_{n+i}$ $i=0,\frac{1}{3},1$ and collocating the third order derivative of (25) at $x=x_{n+i}$, $i=0,\frac{1}{3},1,2$ yields a system of 7 equations each of degree six (i.e., r+s-1=6) as follows:

$$\begin{pmatrix} 1 & -\frac{23}{7} & \frac{43}{3} & -\frac{34239}{491} & \frac{11042}{31} & -\frac{20577}{11} & \frac{400121}{40} \\ 1 & -\frac{18}{7} & \frac{4363}{513} & -\frac{5537}{176} & \frac{20555}{168} & -\frac{12236}{25} & \frac{61811}{31} \\ 1 & -\frac{8}{7} & \frac{67}{57} & -\frac{584}{491} & \frac{5221}{4361} & -\frac{9496}{7899} & \frac{87419}{72509} \\ 0 & 0 & 0 & \frac{64050}{491} & -\frac{30346}{11} & \frac{258619}{7} & -400214 \\ 0 & 0 & 0 & \frac{64050}{491} & -\frac{96673}{45} & \frac{155489}{7} & -184945 \\ 0 & 0 & 0 & \frac{64050}{491} & -\frac{20403}{22} & \frac{191480}{51} & -\frac{56859}{5} \\ 0 & 0 & 0 & \frac{64050}{491} & \frac{80448}{89} & 3586 & \frac{85491}{8} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} h^3 f_k \\ h^3 f_{k+1/3} \\ h^3 f_{k+1} \\ h^3 f_{k+2} \\ y_k \\ y_{k+1/3} \\ y_{k+1} \end{pmatrix} \qquad (26)$$

Equation (26) is solved by using MATLAB software to obtain the values of the unknown parameters a_n's=0(1)6, as follows:

$$a_0 = -\frac{37}{3158}h^3 f_k + \frac{202}{2215}h^3 f_{k+\frac{1}{3}} + \frac{443}{3496}h^3 f_{k+1} + \frac{35}{22133}h^3 f_{k+2} + \frac{13}{6}y_k - \frac{81}{20}y_{k+\frac{1}{3}} + \frac{173}{60}y_{k+1}$$

$$a_1 = -\frac{54}{6493}h^3 f_k + \frac{331}{3676}h^3 f_{k+\frac{1}{3}} + \frac{553}{3215}h^3 f_{k+1} + \frac{90}{20431}h^3 f_{k+2} + \frac{1316}{555}y_k - \frac{315}{74}y_{k+\frac{1}{3}} + \frac{2093}{1110}y_{k+1}$$

$$a_2 = \frac{225}{48149}h^3 f_k + \frac{35}{10183}h^3 f_{k+\frac{1}{3}} + \frac{599}{8667}h^3 f_{k+1} + \frac{101}{21259}h^3 f_{k+2} + \frac{171}{370}y_k - \frac{513}{740}y_{k+\frac{1}{3}} + \frac{171}{740}y_{k+1}$$

$$a_3 = \frac{31}{10352}h^3 f_k - \frac{87}{12566}h^3 f_{k+\frac{1}{3}} + \frac{79}{8464}h^3 f_{k+1} + \frac{34}{15037}h^3 f_{k+2}$$

$$a_4 = \frac{18}{129337}h^3 f_k - \frac{25}{101836}h^3 f_{k+\frac{1}{3}} - \frac{13}{31307}h^3 f_{k+1} + \frac{16}{30677}h^3 f_{k+2}$$

$$a_5 = -\frac{7}{65333}h^3 f_k + \frac{12}{48511}h^3 f_{k+\frac{1}{3}} - \frac{4}{20203}h^3 f_{k+1} + \frac{9}{155797}h^3 f_{k+2}$$

$$a_6 = -\frac{3}{242467}h^3 f_k + \frac{3}{134704}h^3 f_{k+\frac{1}{3}} - \frac{3}{242467}h^3 f_{k+1} + \frac{1}{404112}h^3 f_{k+2} \qquad (27)$$

Substituting (27) into (25) yields a continuous implicit hybrid two –step method in the form :

$$y(x) = \alpha_0(t)y_n + \alpha_{\frac{1}{3}}(t)y_{n+\frac{1}{3}} + \alpha_1(t)y_{n+1} + h^3(\beta_0(t)y_n + \beta_{\frac{1}{3}}(t)y_{n+\frac{1}{3}} + \beta_1(t)y_{n+1} + \beta_2(t)y_{n+2}) \qquad (28)$$

where

$$\alpha_0(t) = 3t^2 + 2t$$

$$\alpha_{\frac{1}{3}}(t) = -\frac{9}{2}(t^2 + t)$$

$$\alpha_1(t) = \frac{1}{2}(3t^2 + 5t + 2)$$

$$\beta_0(t) = h^3\left(-\frac{1}{80}t^6 + \frac{1}{120}t^5 - \frac{1}{24}t^4 + \frac{1}{447}t^3 - \frac{229}{6480}t^2 - \frac{47}{3240}t + \frac{1}{834}\right) \qquad (29)$$

$$\beta_{\frac{1}{3}}(t) = h^3\left(\frac{9}{400}t^6 - \frac{1}{313}t^5 - \frac{9}{80}t^4 + \frac{1}{193}t^3 + \frac{161}{900}t^2 + \frac{4}{45}t - \frac{1}{179}\right)$$

$$\beta_1(t) = h^3\left(-\frac{1}{80}t^6 - \frac{1}{60}t^5 + \frac{1}{16}t^4 + \frac{1}{6}t^3 + \frac{56}{405}t^2 + \frac{31}{810}t - \frac{1}{516}\right)$$

$$\beta_2(t) = h^3\left(\frac{1}{400}t^6 + \frac{1}{120}t^5 + \frac{1}{120}t^4 - \frac{1}{791}t^3 - \frac{67}{16571}t^2 - \frac{1}{648}t - \frac{1}{774}\right)$$

and $t = \frac{x - x_n - h}{h}$.

Evaluating equation (28) at $x = x_{n+2}$ yields the discrete equation:

$$y_{n+2} = 5y_n - 9y_{n+\frac{1}{3}} + 5y_{n+1} + \frac{h3}{810}\left(-10f_n + 144f_{n+\frac{1}{3}} + 305f_{n+1} + 11f_{n+2}\right) \quad (30)$$

The scheme is of order $P = 4$; and the error constant is $C_{p+3} = -\frac{5}{11664}$

The first derivatives of continuous functions are given as:

$$\alpha_0^{'}(t) = \frac{1}{h}(6t + 2)$$

$$\alpha_{\frac{1}{3}}^{'}(t) = -\frac{1}{h}\left(9t + \frac{9}{2}\right)$$

$$\alpha_1^{'}(t) = \frac{1}{h}\left(3t + \frac{5}{2}\right)$$

$$\beta_0^{'}(t) = h^2\left(-\frac{3}{40}t^5 + \frac{1}{24}t^4 + \frac{1}{6}t^3 + \frac{1}{746}t^2 - \frac{229}{3240}t - \frac{47}{3240}\right) \quad (31)$$

$$\beta_{\frac{1}{3}}^{'}(t) = h^2\left(\frac{27}{200}t^5 + \frac{1}{121}t^4 - \frac{9}{20}t^3 + \frac{1}{285}t^2 + \frac{161}{450}t + \frac{4}{45}\right)$$

$$\beta_1^{'}(t) = h^2\left(-\frac{3}{40}t^5 - \frac{1}{12}t^4 + \frac{1}{4}t^3 + \frac{1}{2}t^2 + \frac{112}{405}t + \frac{31}{810}\right)$$

$$\beta_2^{'}(t) = h^2\left(\frac{3}{200}t^5 + \frac{1}{24}t^4 + \frac{1}{30}t^3 + \frac{1}{408}t^2 - \frac{131}{16200}t - \frac{1}{648}\right)$$

where the differentiation is w.r.t. the variable x. The second derivatives of continuous function are given as:

$$\alpha_0^{''}(t) = \frac{6}{h^2}$$

$$\alpha_{\frac{2}{3}}^{''}(t) = -\frac{9}{h^2}$$

$$\alpha_1^{''}(t) = \frac{3}{h^2}$$

$$\beta_0^{''}(t) = h\left(-\frac{3}{8}t^4 + \frac{1}{6}t^3 + \frac{1}{2}t^2 + \frac{1}{373}t - \frac{229}{3240}\right) \quad (32)$$

$$\beta_{\frac{1}{3}}^{''}(t) = h\left(\frac{27}{40}t^4 + \frac{1}{303}t^3 - \frac{27}{20}t^2 + \frac{1}{142}t + \frac{161}{450}\right)$$

$$\beta_1^{''}(t) = h\left(-\frac{3}{8}t^4 - \frac{1}{3}t^3 + \frac{3}{4}t^2 + t + \frac{112}{405}\right)$$

$$\beta_2^{''}(t) = h\left(\frac{3}{40}t^4 + \frac{1}{6}t^3 + \frac{1}{10}t^2 + \frac{1}{204}t - \frac{131}{16200}\right)$$

Equation (31) and (32) when evaluated at x_k, $x_{k+1/3}$, x_{k+1} and x_{k+2} respectively yields the following discrete derivative schemes:

$$hy'_k + 4y_k - \frac{9}{2}y_{k+\frac{1}{3}} + \frac{1}{2}y_{k+1} = h^3\left(\frac{1}{162}f_k + \frac{83}{1800}f_{k+\frac{1}{3}} + \frac{11}{3240}f_{k+1} - \frac{1}{8100}f_{k+2}\right) (33)$$

$$hy'_{k+\frac{1}{3}} + 2y_k - \frac{3}{2}y_{k+\frac{1}{3}} - \frac{1}{2}y_{k+1} = h^3\left(\frac{13}{9720}f_k - \frac{23}{675}f_{k+\frac{1}{3}} - \frac{11}{2430}f_{k+1} + \frac{11}{48600}f_{k+2}\right) (34)$$

$$hy'_{k+1} - 2y_k + \frac{9}{2}y_{k+\frac{1}{3}} - \frac{5}{2}y_{k+1} = h^3\left(-\frac{47}{3240}f_k + \frac{4}{45}f_{k+\frac{1}{3}} + \frac{31}{810}f_{k+1} - \frac{1}{648}f_{k+2}\right) (35)$$

$$hy'_{k+2} - 8y_k + \frac{27}{2}y_{k+\frac{1}{3}} - \frac{11}{2}y_{k+1} = h^3\left(\frac{13}{270}f_k + \frac{79}{600}f_{k+\frac{1}{3}} + \frac{979}{1080}f_{k+1} + \frac{217}{2700}f_{k+2}\right) (36)$$

$$h^2y''_k - 6y_k + 9y_{k+\frac{1}{3}} - 3y_{k+1} = h^3\left(-\frac{91}{810}f_k - \frac{571}{1800}f_{k+\frac{1}{3}} - \frac{49}{3240}f_{k+1} + \frac{1}{4050}f_{k+2}\right) \quad (37)$$

$$h^2y''_{k+\frac{1}{3}} - 6y_k + 9y_{k+\frac{1}{3}} - 3y_{k+1} = h^3\left(\frac{91}{3240}f_k - \frac{49}{450}f_{k+\frac{1}{3}} - \frac{13}{405}f_{k+1} + \frac{29}{16200}f_{k+2}\right) \quad (38)$$

$$h^2y''_{k+1} - 6y_k + 9y_{k+\frac{1}{3}} - 3y_{k+1} = h^3\left(-\frac{229}{3240}f_k + \frac{161}{450}f_{k+\frac{1}{3}} + \frac{112}{405}f_{k+1} - \frac{131}{16200}f_{k+2}\right) (39)$$

$$h^2y''_{k+2} - 6y_k + 9y_{k+\frac{1}{3}} - 3y_{k+1} = h^3\left(\frac{179}{810}f_k - \frac{571}{1800}f_{k+\frac{1}{3}} + \frac{2121}{1609}f_{k+1} + \frac{450}{1349}f_{k+2}\right) (40)$$

Equations (30), (33)-(40) are solved using Shampine and Watts (1969) block formula defined as:

$$Ay_m = hBF(y_m) + ey_n + hdf_n \qquad (41)$$

where $A = (a_{ij})$, $B = (b_{ij})$, column vectors

$e = (e_1 ... e_r)^T$, $d = (d_1 ... d_r)^T$

$y_m = (y_{n+1} ... y_{n+r})^T$ and

$F(y_m) = (f_{n+1} ... f_{n+r})^T$.

According to (33)-(40), A, B, d and e are obtained from Shampine Equation (41) as follows:

$$A=\begin{pmatrix} 9 & -5 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ -\frac{9}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -\frac{3}{2} & -\frac{1}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ \frac{9}{2} & -\frac{5}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ \frac{27}{2} & -\frac{11}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 9 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 9 & -3 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 9 & -3 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 9 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B=\begin{pmatrix} \frac{8}{45} & \frac{61}{162} & \frac{11}{810}\\ \frac{83}{1800} & \frac{11}{3240} & -\frac{1}{8100}\\ -\frac{23}{675} & -\frac{11}{2430} & \frac{11}{48600}\\ \frac{4}{45} & \frac{31}{810} & -\frac{1}{648}\\ \frac{79}{600} & \frac{979}{1080} & \frac{217}{2700}\\ -\frac{571}{1800} & -\frac{49}{3240} & \frac{1}{4050}\\ -\frac{49}{450} & -\frac{13}{405} & \frac{29}{16200}\\ \frac{161}{450} & \frac{112}{405} & -\frac{131}{16200}\\ -\frac{571}{1800} & \frac{2121}{1609} & \frac{450}{1349} \end{pmatrix} \qquad E=\begin{pmatrix} 5 & 0 & 0\\ -4 & -1 & 0\\ -2 & 0 & 0\\ 2 & 0 & 0\\ 8 & 0 & 0\\ 6 & 0 & -1\\ 6 & 0 & 0\\ 6 & 0 & 0\\ 6 & 0 & 0 \end{pmatrix}$$

$$D=\begin{pmatrix} -\frac{1}{81} & \frac{1}{162} & \frac{13}{9720} & -\frac{47}{3240} & \frac{13}{270} & -\frac{91}{810} & \frac{91}{3240} & -\frac{229}{3240} & \frac{179}{810} \end{pmatrix}^T$$

Substituting A , B ,D and E into (41), we obtain the explicit schemes:

$$
\left.
\begin{aligned}
y_{k+\frac{1}{3}} &= y_k + \frac{1}{3}hy_k' + \frac{1}{18}h^2y_k'' + \frac{61}{14580}h^3f_k + \frac{73}{32400}h^3f_{k+\frac{1}{3}} - \frac{17}{58320}h^3f_{k+1} + \frac{1}{36450}h^3f_{k+2}\\
y_{k+1} &= y_k + hy_k' + \frac{1}{2}h^2y_k'' + \frac{1}{20}h^3f_k + \frac{9}{80}h^3f_{k+\frac{1}{3}} + \frac{1}{240}h^3f_{k+1}\\
y_{k+2} &= y_k + 2hy_k' + 2h^2y_k'' + \frac{1}{5}h^3f_k + \frac{18}{25}h^3f_{k+\frac{1}{3}} + \frac{2}{5}h^3f_{k+1} + \frac{1}{75}h^3f_{k+2}\\
y_{k+\frac{1}{3}}' &= hy_k' + \frac{1}{3}h^2y_k'' + \frac{317}{9720}h^3f_k + \frac{23}{900}h^3f_{k+\frac{1}{3}} - \frac{7}{2430}h^3f_{k+1} + \frac{13}{48600}h^3f_{k+2}\\
y_{k+1}' &= hy_k' + h^2y_k'' + \frac{11}{120}h^3f_k + \frac{9}{25}h^3f_{k+\frac{1}{3}} + \frac{1}{20}h^3f_{k+1} - \frac{1}{600}h^3f_{k+2}\\
y_{k+2}' &= hy_k' + 2h^2y_k'' + \frac{4}{15}h^3f_k + \frac{18}{25}h^3f_{k+\frac{1}{3}} + \frac{14}{15}h^3f_{k+1} + \frac{2}{25}h^3f_{k+2}\\
y_{k+\frac{1}{3}}'' &= h^2y_k'' + \frac{91}{648}h^3f_k + \frac{5}{24}h^3f_{k+\frac{1}{3}} - \frac{11}{648}h^3f_{k+1} + \frac{1}{648}h^3f_{k+2}\\
y_{k+1}'' &= h^2y_k'' + \frac{1}{24}h^3f_k + \frac{27}{40}h^3f_{k+\frac{1}{3}} + \frac{7}{24}h^3f_{k+1} - \frac{1}{120}h^3f_{k+2}\\
y_{k+2}'' &= h^2y_k'' + \frac{1}{3}h^3f_k + \frac{4}{3}h^3f_{k+1} + \frac{1}{3}h^3f_{k+2}
\end{aligned}
\right\}(42)
$$

The block formulae are all of order, p=4 with the error constants:

$$C_{P+3} = \left[-\frac{53}{7348320}, -\frac{1}{30240}, -\frac{1}{1890}, -\frac{73}{1049760}, \frac{1}{4320}, -\frac{1}{270}, -\frac{23}{58320}, \frac{1}{720}, -\frac{1}{90}\right]^T$$, respectively.

ANALYSIS OF THE METHOD

The basic properties of this method such as order, error constant, zero stability and consistency are analyzed hereunder. Equation (xv) derived is a discrete scheme belonging to the class of LMMs of the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{43}$$

Following Fatunla (1988) and Lambert (1973), we define the local truncation error associated with Equation (43) by the difference operator:

$$L[y(x):h] = \sum_{j=0}^{k} \left[\alpha_j y(x_n + jh) - h^3 \beta_j f(x_n + jh)\right] \tag{44}$$

Where $y(x)$ is an arbitrary function, continuously differentiable on [a, b]. Expanding (44) in Taylor series about the point x, we obtain the expression:

$$L[y(x);h] = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + \ldots + C_{p+3} h^{p+3} y^{p+3}(x)$$

Where the C_0, C_1, $C_2 \dots C_p \dots C_{p+2}$ are obtained as :

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j$$

$$C_2 = \frac{1}{2!}\sum_{j=1}^{k} j^2\alpha_j$$

$$C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q\alpha_j - q(q-1)(q-2)\sum_{j=1}^{k}\beta_j j^{q-3}\right]$$

In the sense of Lambert (1973), Equation (44) is of order P if :

$C_0 = C_1 = C_2 = \dots C_p = C_{p+2} = 0$ and $C_{p+3} \neq 0$

The $C_{p+3} \neq 0$ is called the error constant and $C_{p+3}h^{p+3}y^{p+3}(x_n)$ is the principal local truncation error at the point x_n.

Zero Stability of the Method

The linear multistep method (43) is said to be zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the method, we present (42) in vector notation form of column vectors $e = (e_1 \dots e_r)^T$, $d = (d_1 \dots d_r)^T$, $y_m = (y_{n+1} \dots y_{n+r})^T$, $F(y_m) = (f_{n+1} \dots f_{n+r})^T$ and matrices $A = (a_{ij})$, $B = (b_{ij})$.

Thus, Equation (42) forms the block formula

$$A^0 y_m = hBF(y_m) + A^1 y_n + hdf_n \qquad (45)$$

where h is a fixed mesh size within a block. In line with (45) and block method (42):

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{k+\frac{1}{3}} \\ y_{k+1} \\ y_{k+2} \end{pmatrix},$$

$$A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{k-2} \\ y_{k-1} \\ y_k \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{73}{32400} & -\frac{17}{58320} & \frac{1}{36450} \\ \frac{9}{80} & \frac{1}{240} & 0 \\ \frac{18}{25} & \frac{2}{5} & \frac{1}{75} \end{pmatrix}\begin{pmatrix} f_{k+\frac{1}{3}} \\ f_{k+1} \\ f_{k+2} \end{pmatrix},$$

$$d = \begin{pmatrix} \frac{61}{14580} \\ \frac{1}{20} \\ \frac{1}{5} \end{pmatrix}\begin{pmatrix} f_{k-2} \\ f_{k-1} \\ f_k \end{pmatrix}$$

The first characteristic polynomial of the block hybrid method is given by:

$$\rho(R) = \det(RA^0 - A^1) \qquad (46)$$

where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Substituting A^0 and A^1 in Equation (45) and solving for R, the values of R are obtained as 0 and 1.

According to Fatunla (1988, 1991), the block method Equation (42) are zero-stable, since from

(46), $\rho(R)=0$, satisfy $|R_j|\le 1$, $j=1$ and for those roots with $|R_j|=1$, the multiplicity does not exceed three.

Consistency of the Method

The linear multistep method(43) is said to be consistent if it has order $P\ge 1$ and the first and second characteristic polynomials which are defined as $\rho(R)=\sum_{j=0}^{k}\alpha_j R^j$ and

$$\sigma(R)=\sum_{j=0}^{k}\beta_j R^j$$

where R, the principal root satisfies the following conditions:

(i) $\sum_{j=0}^{k}\alpha_j=0$, (ii) $\rho(1)=\rho'(1)=0$, (iii) $\rho'''(1)=3!\sigma(1)$

The hybrid scheme (33) derived is of order $P=4>1$ and it has been investigated to satisfy conditions (i)…(iii). Hence the scheme is consistent.

Convergency of the Method

According to the theorem of Dahlquist, the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions hence it is convergent.

Numerical Examples

We consider here four test problems to illustrate the method.

Problem 1: (A constant coefficient homogeneous problem)

$$y'''+y'=0$$
$$y(0)=0,\quad y'(0)=1,\quad y''(0)=2$$

Exact solution:

$$y(x)=2(1-Cosx)+Sinx$$

Problem 2: (A constant coefficient non homogeneous problem)

$$y'''+4y'=x$$
$$y(0)=y'(0)=0,\quad y''(0)=1,\quad h=0.1,\quad 0\le x\le 1$$

Exact solution:

$$y(x)=\frac{3}{16}(1-\cos 2x)+\frac{1}{8}x^2$$

Problem 3: (A variable coefficient singular problem)

$$y'''+\frac{\cos x}{\sin x}y''=\sin x\cos x$$
$$y(0)=1,\ y'(0)=-2,\quad y''(0)=0,\ h=0.1$$

Exact solution:

$$y(x)=1-2x+\frac{x^2}{12}-\frac{\sin^2 x}{12}$$

Problem 4: (A non-linear non- homogeneous problem)

$$y'''=y'(2xy''+y'),\quad y(0)=1,$$
$$y'(0)=\frac{1}{2}\qquad y''(0)=0,$$
$$h=0.01$$

Exact solution:

$$y(x)=1+\frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$$

Table 1: Comparison of the New Block Method and Anake Block Algorithm for Problem 1.

X	Exact Solution	Results of the PM	Error	Error in Anake
0.1	0.10982508609077662011	0.10982508608720526482	3.57135529×10^{-12}	1.60880000×10^{-9}
0.2	0.23853617511257795326	0.23853617521272132138	$1.0014336812\times10^{-10}$	1.03870000×10^{-8}
0.3	0.38484722841012753581	0.38484722889856993877	$4.8844240296\times10^{-10}$	2.95720000×10^{-8}
0.4	0.54729635430288032607	0.54729635566304631116	$1.36016598509\times10^{-09}$	2.31470000×10^{-7}
0.5	0.72426041482345756807	0.72426041775514166541	$2.93168409734\times10^{-09}$	4.54200000×10^{-7}
0.6	0.91397124357567876270	0.91397124900816441044	$5.43248564774\times10^{-09}$	1.47460000×10^{-6}
0.7	1.11453331266871420120	1.11453332176914982600	$9.1004356248\times10^{-09}$	2.87340000×10^{-6}
0.8	1.32394267220519191980	1.32394268638196401810	$1.41767720983\times10^{-08}$	4.68260000×10^{-6}
0.9	1.54010697308615447550	1.54010699398707376820	$2.09009192927\times10^{-08}$	6.92170000×10^{-6}
1.0	1.76086637307161707180	1.76086640257681337610	$2.95051963043\times10^{-08}$	9.59740000×10^{-6}

Table 2: Comparison of the New Block Method and Anake Block Algorithm for Problem 2.

X	Exact Solution	Results of the PM	Error	Error in Anake
0.1	0.00498751665476719416	0.00498751664825035050	$6.5168436600\times10^{-12}$	$2.0952x10^{-09}$
0.2	0.01980106362445904698	0.01980106397185038989	$-3.4739134291\times10^{-10}$	$1.6375x10^{-08}$
0.3	0.04399957220443531927	0.04399957422602327400	$-2.02158795473\times10^{-09}$	$1.1154x10^{-07}$
0.4	0.07686749199740648358	0.07686749852682062271	$-6.52941413913\times10^{-09}$	$9.8800x10^{-07}$
0.5	0.11744331764972380299	0.11744333346408755609	$-1.58143637531\times10^{-08}$	$3.0406x10^{-06}$
0.6	0.16455792103562370419	0.16455795312358073792	$-3.208795703373\times10^{-08}$	$9.0126x10^{-06}$
0.7	0.21688116070620482401	0.21688121834485580696	$-5.763865098295\times10^{-08}$	$1.6965x10^{-05}$
0.8	0.27297491043149163616	0.27297500505499946023	$-9.462350782407\times10^{-08}$	$2.6772x10^{-05}$
0.9	0.33135039275495382287	0.33135053761054919072	$-1.4485559536785\times10^{-07}$	$3.8135x10^{-05}$
1.0	0.39052753185258919756	0.39052774145323146804	$-2.0960064227048\times10^{-07}$	$5.0596x10^{-05}$

Table 3: Comparison of the New Block Method and Falade Thesis for Problem 3.

X	Exact Solution	Results of the PM	Error	Error in Falade
0.1	0.80000277407671840129	0.80000277407319636487	$3.5220364200\times10^{-12}$	0.5789E-06
0.2	0.60004420808345354511	0.60004420800195996358	$8.14935815300\times10^{-11}$	0.3806 E-05
0.3	0.40022231728790326238	0.40022231690635359165	$3.815496707300\times10^{-10}$	0.1037 E-04
0.4	0.20069611288946522587	0.20069611174898726099	$1.1404779648800\times10^{-09}$	0.2341 E-04
0.5	0.00167926274450582156	0.00167926006910235206	$2.675403469498\times10^{-09}$	0.4126 E-04
0.6	-0.1965684268968052676	-0.1965684322683223386	$5.3715170710300\times10^{-09}$	0.6591 E-04
0.7	-0.3937513690458232942	-0.3937513787126569529	$9.6668336586800\times10^{-09}$	0.9904 E-02
0.8	-0.5895499800958870302	-0.5895499961306756554	$1.6034788625200\times10^{-08}$	0.1432 E-03
0.9	-0.7836334206122119606	-0.7836334455776121717	$2.4965400211060\times10^{-08}$	0.2018 E-03
1.0	-0.97567278485613093	-0.9756728214463478494	$3.65902169166300\times10^{-08}$	0.7197 E-03

Table 4: Comparison of the New Methods with Existing Method.

X	Exact Solution	Results of the PM	Error	Error in Olabode
0.21	1.10538844783849891010	1.10538844783837950530	$1.194048000\times10^{-13}$	$3.157979012E-08$
0.31	1.15625949779936003420	1.15625949779895135000	$4.0868420000\times10^{-13}$	$9.636302289E-08$
0.41	1.20794636563521173500	1.20794636563419504550	$1.0166895000\times10^{-12}$	$2.640700834E-07$
0.51	1.26075331659316237260	1.26075331659102288900	$2.1394836000\times10^{-12}$	$6.260533061E-07$
0.61	1.31502323709600082640	1.31502323709191724620	$4.0835802000\times10^{-12}$	$1.348230303E-06$
0.71	1.37115320825901439260	1.37115320825166432330	$7.3500693000\times10^{-12}$	$2.698695479E-06$
0.81	1.42961558811110818990	1.42961558809831614740	$1.2792042500\times10^{-11}$	$7.814267388E-06$

CONCLUSION

The continuous implicit two-step hybrid method all of order four have been developed by the interpolation and collocation technique for the approximation of the solution of initial value problems in third order ordinary differential equation.

The scheme is in the block form and consequently they do not require other method (especially one-step methods) in order to implement them.

The methods were applied to four problems, each with its own peculiarity and the results thereby obtained demonstrate their effectiveness and accuracy viz-a-viz some other existing schemes (Anake (2013), Awoyemi(2014)), as they perform favorably well.

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