

# On Integration of $y'' = f(x, y, y')$ using New Class of Orthogonal Polynomials as Trial Function.

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## ABSTRACT

We propose continuous two-step hybrid linear multistep method (CTHLMM) through collocation technique for direct integration of general second order initial value problems. Our keen interest is to construct a class of orthogonal polynomials with recursive formula which shall produce the same results as existing polynomials. The basic properties of the method such as order, stability, consistency and convergence are discussed. The CTHLMM is implemented on four test problems and on comparison with the exact solutions and the existing method, efficiency and accuracy of CTHLMM are established.

(Keyword: continuous two-step hybrid linear multistep method, CTHLMM)

## INTRODUCTION

We focus on formulation of a continuous hybrid two-step linear multistep method (CHTLMM):

$$\sum_{i=0}^k \alpha_i y_{n+i} = h^2 \left[ \sum_{i=0}^k \beta_i f_{n+i} + \sum_{i=1}^k \beta_i f_{n+w} \right],$$

$$\alpha_k = 1, \beta_k \neq 0$$

for the numerical solution of second order initial value problem (IVP):

$$\begin{aligned} y''(x) &= f(x, y(x), y'(x)) & y(x_0) &= y_0, \\ y'(x_0) &= y'_0, & p \leq x \leq q \end{aligned} \quad (1)$$

where,

$$f : \mathcal{R} \times \mathcal{R}^{nm} \rightarrow \mathcal{R}^m, y : \mathcal{R} \rightarrow \mathcal{R}^m, x_0 \leq x \leq x_{end},$$

$n = 2$  and  $m$  is the dimension of the system.

The desire for the solution of (1) is owing to its occurrence in several areas of engineering, management and science, such as celestial mechanics, circuit theory, control theory, chemical kinetics, and biology to mention but a few. It is commonly reported that (1) is difficult to solve or has no solution analytically hence the need for numerical methods.

Researchers have developed several methods with various techniques to directly obtain the solution of (1). Among them are the linear multistep methods (LMMs) (see Lambert and Watson (1976), Henrici (1962), Stiefel, and Bettis (1969)), multistep collocation methods (see Carpentieri and Paternoster (2005), D'Ambrosio et al. (2009), Coleman and Duxbury (2000)) and multi-derivative methods (Twizell and Khaliq (1984)).

The approach of reducing (1) to a system of first order equation has also been reported in the literature to increase the dimension of the problem, lead to more computation work, waste a lot of computer time and and human efforts (see Lambert (1973), Awoyemi (1991, 1999), Fatunla (1988), Jator (2010)).

To cater for this setback, Milne (1953) proposed block method which was later developed into algorithm by Rosser (1967) to implement (1) directly. The approach has been widely used by scholars. However, to derive all the aforementioned methods and several others, polynomials play a vital role. Notable among the well-known polynomials are the power series, Legendre polynomials, Chebyshev polynomials, the general Jacobi polynomials, the Hermite and the Laguerre polynomials and canonical polynomials.

In this work, our keen interest is to construct a new class of orthogonal polynomials and employ same to formulate CTHLMM which shall produce

the same results as the other existing polynomials.

In what follows, we shall construct a set of polynomials valid in interval  $[-1, 1]$  with respect to weight function  $w(x) = x^2 - 1$ .

### CONSTRUCTION OF THE BASIS FUNCTION

Let the function  $q_n(x)$ , the quantity to be evaluated be defined as:

$$q_r(x) = \sum_{r=0}^n C_r^{(n)} x^r \quad (2a)$$

on the real interval  $[a, b]$  where  $q_r(x)$  must satisfy the orthogonal property:

$$\begin{aligned} \langle q_m(x), q_n(x) \rangle &= \int_a^b w(x) q_m(x) q_n(x) dx \\ &= 0, \quad m \neq n \end{aligned}$$

(2b)

For the purpose of constructing the basis function, we adopt the approach discussed extensively in Adeyefa and Adeniyi (2015) and use additional property (the normalization)  $q_n(1)=1$  where our weight function is defined as  $w(x) = x^2 - 1$ .

Defining (2a) over the interval  $[-1, 1]$  and using (2) yields:

$$\left. \begin{aligned} q_0(x) &= 1 \\ q_1(x) &= x \\ q_2(x) &= \frac{1}{4}(5x^2 - 1) \\ q_3(x) &= \frac{1}{4}(7x^3 - x) \\ q_4(x) &= \frac{1}{8}(21x^4 - 14x^2 + 1) \\ q_5(x) &= \frac{1}{8}(33x^5 - 30x^3 + 5x) \\ q_6(x) &= \frac{1}{64}(429x^6 - 495x^4 + 135x^2 - 5) \\ q_7(x) &= \frac{1}{64}(715x^7 - 1001x^5 + 385x^3 - 35x) \\ q_8(x) &= \frac{1}{128}(2431x^8 - 4004x^6 + 2002x^4 - 308x^2 + 7) \\ q_9(x) &= \frac{1}{128}(4199x^9 - 7956x^7 + 4914x^5 - 1092x^3 + 63x) \\ q_{10}(x) &= \frac{1}{512}(29393x^{10} - 62985x^8 + 46410x^6 - 13650x^4 + 1365x^2 - 21) \end{aligned} \right\} (3)$$

In the spirit of Golub and Fischer (1992), equation (3) must satisfy three-term recurrence relation:

$$c_j p(t) = (t - a_j) p_{j-1}(t) - b_j p_{j-2}(t),$$

$$j = 1, 2, \dots, p_{-1}(t) = 0, p_0(t) \equiv p_0$$

where  $b_j, c_j > 0$  for  $j \geq 1$  ( $b_1$  is arbitrary).

$$c_j p(t) = (n + 3) P_{n+1}(x),$$

$$(t - a_j) p_{j-1}(t) = (2n + 3) x P_n(x),$$

$$b_j p_{j-2}(t) = n P_{n-1}(x), n = 1, 2, \dots$$

The recursive formula for this class orthogonal polynomial, say ADEM-B orthogonal polynomial is therefore given as:

$$P_{n+1}(x) = \frac{1}{n+3} [(2n+3)xP_n(x) - nP_{n-1}(x)],$$

$$n \geq 1, P_0(x) = 1, P_1(x) = x$$

In the next section, these polynomials shall be employed to construct the proposed CTHLMM.

## DEVELOPMENT OF THE METHOD

We proposed an approximate solution to (1) in the form:

$$y(x) = \sum_{r=0}^{s+k-1} a_r q_r(x) \tag{4}$$

where  $s$  and  $k$  in (4) are points of interpolation and collocation respectively. Transforming  $q_n(x)$  to interval  $[0, 1]$ , we have  $x = \frac{2X - 2x_n - ph}{ph}$ , where  $p$  varies as the method to be developed. In this case,  $p = 2$ .

The procedure involves interpolating (4) at points  $s = 0, \frac{1}{2}$  and collocating the second derivative of (4) at

points  $k = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 2$ . The  $a_r, r = 0(1)7$  from the resulting system of equations are obtained as:

$$\begin{aligned}
a_0 &= \frac{341}{15120}h^2f_n - \frac{11}{315}h^2f_{n+\frac{1}{4}} + \frac{373}{1080}h^2f_{n+\frac{1}{2}} - \frac{23}{135}h^2f_{n+\frac{3}{4}} + \frac{937}{5040}h^2f_{n+1} + \frac{11}{7560}h^2f_{n+2} - y_n + 2y_{n+\frac{1}{2}} \\
a_1 &= \frac{12343}{166320}h^2f_n - \frac{17261}{72765}h^2f_{n+\frac{1}{4}} + \frac{84913}{83160}h^2f_{n+\frac{1}{2}} - \frac{7531}{10395}h^2f_{n+\frac{3}{4}} + \frac{101609}{166320}h^2f_{n+1} + \\
&\quad \frac{3191}{582120}h^2f_{n+2} - 2y_n + 2y_{n+\frac{1}{2}} \\
a_2 &= \frac{161}{1485}h^2f_n - \frac{2048}{3465}h^2f_{n+\frac{1}{4}} + \frac{2048}{1485}h^2f_{n+\frac{1}{2}} - \frac{2048}{1485}h^2f_{n+\frac{3}{4}} + \frac{48}{55}h^2f_{n+1} + \frac{103}{10395}h^2f_{n+2} \\
a_3 &= \frac{107}{1001}h^2f_n - \frac{37888}{63063}h^2f_{n+\frac{1}{4}} + \frac{3712}{3003}h^2f_{n+\frac{1}{2}} - \frac{1024}{715}h^2f_{n+\frac{3}{4}} + \frac{6112}{9009}h^2f_{n+1} + \frac{1213}{105105}h^2f_{n+2} \\
a_4 &= \frac{151}{2457}h^2f_n - \frac{256}{819}h^2f_{n+\frac{1}{4}} + \frac{256}{351}h^2f_{n+\frac{1}{2}} - \frac{256}{351}h^2f_{n+\frac{3}{4}} + \frac{22}{91}h^2f_{n+1} + \frac{23}{2457}h^2f_{n+2} \\
a_5 &= \frac{14}{1485}h^2f_n - \frac{128}{2079}h^2f_{n+\frac{1}{4}} + \frac{112}{1485}h^2f_{n+\frac{1}{2}} - \frac{128}{7425}h^2f_{n+\frac{3}{4}} - \frac{68}{1485}h^2f_{n+1} + \frac{274}{51975}h^2f_{n+2} \\
a_6 &= -\frac{256}{19305}h^2f_n + \frac{4096}{45045}h^2f_{n+\frac{1}{4}} - \frac{4096}{19305}h^2f_{n+\frac{1}{2}} + \frac{4096}{19305}h^2f_{n+\frac{3}{4}} - \frac{512}{6435}h^2f_{n+1} + \frac{256}{135135}h^2f_{n+2} \\
a_7 &= -\frac{512}{45045}h^2f_n + \frac{16384}{315315}h^2f_{n+\frac{1}{4}} - \frac{4096}{45045}h^2f_{n+\frac{1}{2}} + \frac{16384}{225225}h^2f_{n+\frac{3}{4}} - \frac{1024}{45045}h^2f_{n+1} + \frac{512}{1576575}h^2f_{n+2}
\end{aligned} \tag{5}$$

Substituting (5) into (4) yields the continuous implicit method:

$$y(x) = \alpha_0(x)y_n + \alpha_{\frac{1}{2}}(x)y_{n+\frac{1}{2}} + h^2(\beta_k(x)f_{n+k}), \quad k = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 2 \tag{6}$$

Evaluating equation (6) at  $x = x_{n+m}$ ,  $m = \frac{1}{4}, \frac{3}{4}, 1, 2$  yields the discrete equations:

$$\left.
\begin{aligned}
y_{n+\frac{1}{4}} &= -\frac{49}{20480}h^2f_n - \frac{181}{6720}h^2f_{n+\frac{1}{4}} - \frac{3}{2560}h^2f_{n+\frac{1}{2}} - \frac{1}{960}h^2f_{n+\frac{3}{4}} + \frac{3}{10240}h^2f_{n+1} \\
&\quad - \frac{1}{430080}h^2f_{n+2} + \frac{1}{2}y_n + \frac{1}{2}y_{n+\frac{1}{2}} \\
y_{n+\frac{3}{4}} &= \frac{131}{61440}h^2f_n + \frac{223}{6720}h^2f_{n+\frac{1}{4}} + \frac{397}{7680}h^2f_{n+\frac{1}{2}} + \frac{7}{960}h^2f_{n+\frac{3}{4}} - \frac{17}{30720}h^2f_{n+1} \\
&\quad - \frac{1}{430080}h^2f_{n+2} - \frac{1}{2}y_n + \frac{3}{2}y_{n+\frac{1}{2}} \\
y_{n+1} &= \frac{1}{240}h^2f_n + \frac{1}{15}h^2f_{n+\frac{1}{4}} + \frac{13}{120}h^2f_{n+\frac{1}{2}} + \frac{1}{15}h^2f_{n+\frac{3}{4}} + \frac{1}{240}h^2f_{n+1} - y_n + 2y_{n+\frac{1}{2}} \\
y_{n+2} &= \frac{19}{420}h^2f_n - \frac{178}{105}h^2f_{n+\frac{1}{4}} + \frac{269}{60}h^2f_{n+\frac{1}{2}} - \frac{62}{15}h^2f_{n+\frac{3}{4}} + \frac{293}{120}h^2f_{n+1} + \frac{19}{420}h^2f_{n+2} - 3y_n + 4y_{n+\frac{1}{2}}
\end{aligned}
\right\} (7)$$

To develop the block method from the continuous scheme, we adopt the general block formula proposed in Shampine and Watts (1969) in the normalized form given as:

$$A^{(0)}Y_m = ey_m + h^{\mu-\lambda}df'(y_m) + h^{\mu-\lambda}bF(y_m) \quad (8)$$

In the spirit of Baker and Keech (1978), a block-by-block method is a method for computing vectors  $Y_1, Y_2, \dots$  in sequence. Equation (8) is applied in a block-by-block fashion.

Evaluating the first derivative of (6) at  $x = x_{n+j}$ ,  $j = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 2$  to obtain the first derivative equations (FDEs). Substituting the resulting FDEs and equation (7) into (8) and solving simultaneously gives a block formulae

$$\begin{aligned}
y_{n+\frac{1}{4}} &= \frac{20017}{1290240}h^2f_n + \frac{715}{28224}h^2f_{n+\frac{1}{4}} - \frac{2509}{161280}h^2f_{n+\frac{1}{2}} + \frac{31}{4032}h^2f_{n+\frac{3}{4}} - \frac{1123}{645120}h^2f_{n+1} \\
&\quad + \frac{107}{9031680}h^2f_{n+2} + \frac{1}{4}y_n' h + y_n \\
y_{n+\frac{1}{2}} &= \frac{361}{10080}h^2f_n + \frac{461}{4410}h^2f_{n+\frac{1}{4}} - \frac{29}{1008}h^2f_{n+\frac{1}{2}} + \frac{11}{630}h^2f_{n+\frac{3}{4}} - \frac{41}{10080}h^2f_{n+1} \\
&\quad + \frac{1}{35280}h^2f_{n+2} + \frac{1}{2}y_n' h + y_n \\
y_{n+\frac{3}{4}} &= \frac{8007}{143360}h^2f_n + \frac{2979}{15680}h^2f_{n+\frac{1}{4}} + \frac{153}{17920}h^2f_{n+\frac{1}{2}} + \frac{15}{448}h^2f_{n+\frac{3}{4}} - \frac{477}{71680}h^2f_{n+1} \\
&\quad + \frac{9}{200704}h^2f_{n+2} + \frac{3}{4}y_n' h + y_n \\
y_{n+1} &= \frac{191}{2520}h^2f_n + \frac{608}{2205}h^2f_{n+\frac{1}{4}} + \frac{16}{315}h^2f_{n+\frac{1}{2}} + \frac{32}{315}h^2f_{n+\frac{3}{4}} - \frac{1}{252}h^2f_{n+1} + \frac{1}{17640}h^2f_{n+2} + y_n' h + y_n \quad (9) \\
y_{n+2} &= \frac{158}{315}h^2f_n - \frac{2816}{2205}h^2f_{n+\frac{1}{4}} + \frac{1376}{315}h^2f_{n+\frac{1}{2}} - \frac{256}{63}h^2f_{n+\frac{3}{4}} + \frac{764}{315}h^2f_{n+1} + \frac{20}{441}h^2f_{n+2} + 2y_n' h + y_n \\
y_{n+\frac{1}{4}}' &= \frac{3881}{46080}hf_n + \frac{599}{2520}hf_{n+\frac{1}{4}} - \frac{221}{1920}hf_{n+\frac{1}{2}} + \frac{1}{18}hf_{n+\frac{3}{4}} - \frac{287}{23040}hf_{n+1} + \frac{3}{35840}hf_{n+2} + y_n' \\
y_{n+\frac{1}{2}}' &= \frac{227}{2880}hf_n + \frac{37}{105}hf_{n+\frac{1}{4}} + \frac{19}{360}hf_{n+\frac{1}{2}} + \frac{1}{45}hf_{n+\frac{3}{4}} - \frac{1}{160}hf_{n+1} + \frac{1}{20160}hf_{n+2} + y_n' \\
y_{n+\frac{3}{4}}' &= \frac{417}{5120}hf_n + \frac{93}{280}hf_{n+\frac{1}{4}} + \frac{129}{640}hf_{n+\frac{1}{2}} + \frac{3}{20}hf_{n+\frac{3}{4}} - \frac{39}{2560}hf_{n+1} + \frac{3}{35840}hf_{n+2} + y_n' \\
y_{n+1}' &= \frac{7}{90}hf_n + \frac{16}{45}hf_{n+\frac{1}{4}} + \frac{2}{15}hf_{n+\frac{1}{2}} + \frac{16}{45}hf_{n+\frac{3}{4}} + \frac{7}{90}hf_{n+1} + y_n' \\
y_{n+2}' &= \frac{47}{45}hf_n - \frac{512}{105}hf_{n+\frac{1}{4}} + \frac{512}{45}hf_{n+\frac{1}{2}} - \frac{512}{45}hf_{n+\frac{3}{4}} + \frac{28}{5}hf_{n+1} + \frac{73}{315}hf_{n+2} + y_n'
\end{aligned}$$

Equation (9) is our desired block method of which its basic properties shall be discussed in the next section.

## ANALYSIS OF THE CTHLMM

### Order and Error Constant

Following Henrici (1962), the approach adopted in Fatunla (1991, 1994) and Lambert (1973), we define the local truncation error associated with equation (9) by the difference operator:

$$L[y(x); h] = \sum_{j=0}^k [\alpha_j y(x_n + jh) - h^2 \beta_j f(x_n + jh)] \quad (10)$$

where  $y(x)$  is an arbitrary function, continuously differentiable on  $[a, b]$ . Expanding (10) in Taylor series about the point  $x$ , we obtain the expression:

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \dots + C_{p+3} h^{p+3} y^{(p+3)}(x)$$

where the  $C_0, C_1, C_2, C_p, \dots$  are obtained as:

$$C_0 = \sum_{j=0}^k \alpha_j, \quad C_1 = \sum_{j=1}^k j \alpha_j, \quad C_2 = \frac{1}{2!} \sum_{j=1}^k j^2 \alpha_j,$$

$$C_q = \frac{1}{q!} \left[ \sum_{j=1}^k j^q \alpha_j - q(q-1)(q-2) \sum_{j=1}^k \beta_j j^{q-3} \right]$$

According to Lambert (1973), equations (7) and (9) are of order  $p$  if:

$$C_0 = C_1 = C_2 = \dots = C_p = C_{p+1} = 0 \text{ and } C_{p+2} \neq 0$$

The  $C_{p+2} \neq 0$  is called the error constant and  $C_{p+2} h^{p+2} y^{(p+2)}(x_n)$  is the principal local truncation error at the point  $x_n$ .

Thus, equations (7) and (9) are all of order 6 with the error constants:

$$C_{p+2} = \left[ \begin{array}{cccc} \frac{347}{7927234560} & \frac{-1}{30965760} & \frac{-19}{528482304} & \frac{-143}{1032192} \end{array} \right]^T$$

and,

$$C_{p+2} = \left[ \begin{array}{cccc} \frac{-1637}{7927234560} & \frac{-31}{61931520} & \frac{-33}{41943040} & \frac{-1}{967680} \\ \frac{-17}{120960} & \frac{-143}{99090432} & \frac{-29}{30965760} & \\ \frac{-5}{3670016} & \frac{-1}{1935360} & \frac{-5}{12096} & \end{array} \right]^T$$

respectively.

### Zero Stability of the CTHLMM

According to Lambert (1973), a linear multistep method is said to be zero-stable if no root of the first characteristic polynomial  $\rho(R)$  has modulus greater than one and if every root of modulus one has multiplicity not greater than the order of the differential equation.

To analyze the zero-stability of the method, we present (9) in vector notation form of column vectors  $e = (e_1 \dots e_r)^T$ ,  $d = (d_1 \dots d_r)^T$ ,  $y_m = (y_{n+1} \dots y_{n+r})^T$ ,  $F(y_m) = (f_{n+1} \dots f_{n+r})^T$  and matrices  $A = (a_{ij})$ ,  $B = (b_{ij})$ .

Thus, equation (9) forms the block formula:

$$A^0 y_m = hBF(y_m) + A^1 y_n + hdf_n \quad (11a)$$

where  $h$  is a fixed mesh size within a block. The first characteristic polynomial of the hybrid block method (9a) is given by:

$$\rho(R) = \det(RA^0 - A^1) \quad (11b)$$

where:

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{715}{28224} & \frac{-2509}{161280} & \frac{31}{4032} & \frac{-1123}{645120} & \frac{107}{9031680} & 0 & 0 & 0 & 0 & 0 \\ \frac{461}{4410} & \frac{-29}{1008} & \frac{11}{630} & \frac{-41}{10080} & \frac{1}{35280} & 0 & 0 & 0 & 0 & 0 \\ \frac{2979}{15680} & \frac{153}{17920} & \frac{15}{448} & \frac{-477}{71680} & \frac{9}{35280} & 0 & 0 & 0 & 0 & 0 \\ \frac{608}{2205} & \frac{16}{315} & \frac{32}{315} & \frac{-1}{252} & \frac{1}{17640} & 0 & 0 & 0 & 0 & 0 \\ \frac{2205}{-2816} & \frac{1376}{1376} & \frac{-256}{-256} & \frac{764}{764} & \frac{20}{20} & 0 & 0 & 0 & 0 & 0 \\ \frac{599}{2520} & \frac{-221}{1920} & \frac{1}{18} & \frac{-287}{23040} & \frac{3}{35840} & 0 & 0 & 0 & 0 & 0 \\ \frac{37}{105} & \frac{19}{360} & \frac{1}{45} & \frac{-1}{160} & \frac{1}{20160} & 0 & 0 & 0 & 0 & 0 \\ \frac{93}{280} & \frac{129}{640} & \frac{3}{20} & \frac{-39}{2560} & \frac{3}{35840} & 0 & 0 & 0 & 0 & 0 \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-512}{-512} & \frac{512}{512} & \frac{-512}{-512} & \frac{28}{28} & \frac{73}{73} & 0 & 0 & 0 & 0 & 0 \\ \frac{105}{105} & \frac{45}{45} & \frac{45}{45} & \frac{5}{5} & \frac{315}{315} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$



$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{20017}{1290240} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{361}{10080} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{8007}{143360} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{191}{2520} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{158}{315} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3881}{46080} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{227}{2880} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{417}{5120} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{7}{90} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{47}{45} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Substituting  $A^0$  and  $A^1$  in (11b), we obtain  $\rho(R) = R^8(R-1)^2$  which implies that  $R_1 = R_2 \dots = R_8 = 0, R_9 = R_{10} = 1$ .

According to Fatunla (1988, 1991), the our block method equation are zero-stable since from  $\rho(R) = 0$  satisfies  $|R_j| \leq 1, j = 1$  and for those roots with  $|R_j| = 1$ , the multiplicity does not exceed two.

### **Region of Absolute Stability of the Main Methods**

For the region of absolute stability, the following definitions are considered. Given the stability polynomial:

$$\pi(z, \bar{h}) = \rho(z) - \bar{h}\sigma(z) = 0 \tag{12}$$

where  $\bar{h} = h^2 \lambda^2$  and  $\lambda = \frac{df}{dy}$  are assumed constants.

The scheme (7) is said to be absolutely stable if for a given  $\bar{h}$  all the roots  $z_s$  of (12) satisfy  $|z_s| < 1, s=1,2,\dots,n$ , where  $\bar{h} = \lambda h$ .

**Definition:** The region  $\mathfrak{R}$  of the complex  $\bar{h}$ -plane such that the roots of  $\pi(z, \bar{h}) = 0$  lies within the unit circle whenever  $\bar{h}$  lies in the interior of the region is called the region of absolute stability.

**Remark:** Let  $\mathfrak{R}$  be the boundary of the region  $\mathfrak{R}$ . Since the roots of the stability polynomial are continuous functions of  $\bar{h}$ ,  $\bar{h}$  will lie on  $\mathfrak{R}$  when one of the roots of the  $\pi(z, \bar{h})=0$  lies on the boundary of the unit circle. Thus we define (12) in terms of Euler's number,  $\exp i\theta$ , as follows;

$$\pi(\exp(i\theta), h) = \rho(\exp(i\theta)) - \bar{h}\sigma(\exp(i\theta)) = 0 \quad (13)$$

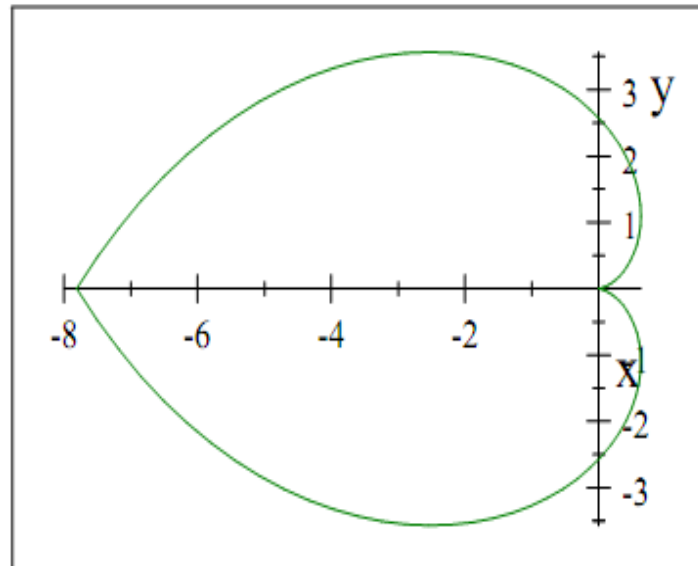
So that, the locus of the boundary  $\mathfrak{R}$  is given by:

$$\bar{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \quad (14)$$

From (7a), the boundary of the region of absolute stability is:

$$\bar{h}(\theta) = \frac{\cos 2\theta + i \sin 2\theta - 4 \cos \frac{1}{2}\theta - 4i \sin \frac{1}{2}\theta + 3}{\frac{19}{420} - \frac{178}{105} \cos \frac{1}{4}\theta - \frac{178}{105} i \sin \frac{1}{4}\theta + \frac{269}{60} \cos \frac{1}{2}\theta + \frac{269}{60} i \sin \frac{1}{2}\theta - \frac{62}{15} \cos \frac{3}{4}\theta - \frac{62}{15} i \sin \frac{3}{4}\theta + \frac{293}{120} \cos \theta + \frac{293}{120} i \sin \theta + \frac{19}{420} \cos 2\theta + \frac{19}{420} i \sin 2\theta}$$

The region of absolute stability (RAS) is shown in Figure 1.



**Figure 1:** RAS of CTHLMM.

### **Consistency of the Method**

According to Lambert (1973), a linear multistep method is said to be consistent if it has order at least one. Owing to this definition, equations (7) and (9) are consistent being of order 6.

### Convergency of the Method

According to the theorem of Dahlquist, the necessary and sufficient condition for a LMM to be convergent is to be consistent and zero stable. Since CTHLMM satisfies the two conditions hence its convergence.

### NUMERICAL EXPERIMENT

#### **Problem 1** (Source: Ramos (2016))

Consider the nonlinear problem given by:

$$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = \frac{1}{2}, h = 0.0025$$

$$\text{Exact Solution : } y(x) = 1 + \frac{1}{2} \ln\left(\frac{2+x}{2-x}\right)$$

#### **Problem 2** (Source: Ramos (2016))

Consider the initial value problem given by:

$$y'' = -\frac{6}{x} y' - \frac{4}{x^2} y, y(1) = 1, y'(1) = 1, h = \frac{0.1}{32}$$

$$\text{Exact Solution : } y(x) = \frac{5x^3 - 2}{3x^4}$$

#### **Problem 3** (Source: Ramos (2016))

$$y'' = y', y(0) = 0, y'(0) = -1, h = 0.1$$

$$\text{Exact Solution : } y(x) = 1 - \exp(x)$$

**Problem 4** We consider in the example the Vanderpol's oscillator problem

$$y'' = 2 \cos x - \cos^3 x - y' - y - y^2 y',$$
$$y(0) = 0, y'(0) = 1, h = 0.1$$

whose analytical solution is  $y(x) = \sin x$

**Table 1:** Comparison of Absolute Errors for Problems 1, 2 and 3.

Problem 1		Problem 2		Problem 3	
CHTLMM, p=6	BFM <sub>6</sub> , p=7	CHTLMM, p=6	BFM <sub>6</sub> , p=7	CHTLMM, p=6	BFM <sub>6</sub> , p=7
1.036e-16	3.11379e-12	1.0e-19	8.816e-18	2.21577e-15	NA
1.88173e-14	6.55987e-12	5.4e-18	2.159e-17	2.6701483e-13	2.427e-11
7.59512e-14	9.833331e-12	2.11e-17	3.346e-17	1.11119883e-12	4.001e-11
1.996016e-13	2.17263e-11	4.14e-17	4.510e-17	2.19958134e-12	5.746e-11
4.579197e-13	3.57048e-11	7.12e-17	5.721e-17	3.69292148e-12	7.741e-11
8.490233e-13	4.85912e-11	1.049e-16	6.497e-17	5.33475937e-12	9.517e-11
1.5073329e-12	1.30979e-10	1.47e-16	7.089e-17	7.13715806e-12	1.221e-10
2.4037899e-12	2.31339e-10	1.926e-16	7.993e-17	8.94144724e-12	1.604e-10
3.7759564e-12	3.28627e-10	2.457e-16	8.829e-17	1.06221891e-11	2.013e-10
5.5675167e-12	1.33465e-9	3.018e-16	9.648e-17	1.213293195e-11	2.466e-10

**Table 2:** Comparison of Exact Solution and CHTLMM for Problem 4.

X	Exact	Approximation	Error
0.1	0.09983341664682815231	0.09983341664682878065	6.28343e-16
0.2	0.19866933079506121546	0.19866933079518066233	1.1944687e-13
0.3	0.29552020666133957511	0.29552020666179674793	4.5717282e-13
0.4	0.38941834230865049167	0.38941834230978998795	1.13949628e-12
0.5	0.47942553860420300027	0.47942553860666947549	2.46647522e-12
0.6	0.56464247339503535720	0.56464247339927031006	4.23495286e-12
0.7	0.64421768723769105367	0.64421768724460349170	6.91243803e-12
0.8	0.71735609089952276163	0.71735609090949508420	9.97232257e-12
0.9	0.78332690962748338846	0.78332690964151488105	1.403149259e-11
1.0	0.84147098480789650665	0.84147098482618172915	1.82852225e-11

## CONCLUSION

An efficient and accurate numerical algorithm based on the collocation and interpolation techniques has been presented with a new class of polynomials as trial function. The CHTLMM conveniently handles both special and general class of second order differential equations. Numerical examples were given to demonstrate the applicability of the algorithm. The results in Table 1 shows that CHTLMM of order  $p=6$  compares favorable well with the  $BFM_6$  of order 7. Table 2 compares the analytical solution and the exact solution.

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## SUGGESTED CITATION

Adeyefa, E.O. 2016. "On Integration of  $y'' = f(x,y,y')$  using New Class of Orthogonal Polynomials as Trial Function". *Pacific Journal of Science and Technology*. 17(2):109-121.

