Integrated Decomposed Fourier Transform Method for Convergence Analysis of Some Hyperbolic Schemes.

S.A. Aniki (Ph.D. in view) and K. Rauf, Ph.D.

Department of Mathematics, University of Ilorin, Kwara State, Nigeria.

E-mail: krauf@unilorin.edu.ng
smlaniki@yahoo.com

ABSTRACT

In this paper, we analyze the stability of some proposed hyperbolic schemes by using Integrated Decomposed Fourier Transform method (IDFTM) which is much more reliable and faster than the conventional Fourier transform approach. On proving stability directly from the definition is quite difficult, in general. Instead it is of high speed to use tools from the IDFTM which shows that the proposed schemes are conditionally stable.

(Keyword: conditionally stable, convergence, decomposition method, Fourier transform, hyperbolic schemes)

INTRODUCTION

Several methods have been developed for the analysis of stability and nearly all of them are limited to linear problems. However, within this restriction the complete investigation for initial and boundary value problem can be extremely complicated particularly in the presence of boundary conditions.

The problem of stability for linear problems with constant coefficient has now become a thing of the past because the influence of boundaries can be removed. This influence in the frequency domain forms the basis for the *Von Neumann* method for stability analysis. This method was developed by *Von Neumann* in Los Alamos during the World War II and was considered classified not until its brief description in Crank and Nicolson and in a publication by Charney [1], [7]. At present, this is the most widely used technique for stability analysis. The problem for stability can be treated generally for linear equations with constant coefficient, in order to deal with the non-linear terms in the basic equations, and then the information on stability becomes very limited. Hence, we have to resort to a local stability analysis, with frozen values of the non-linear coefficients, to make the formulation linear.

Also, Jovanovic and Suli published a paper on Analysis of Finite Difference Schemes, Tadmor on a review of numerical methods for nonlinear partial differential equations, Kent, Jablonowski and Rood on Determining the effective resolution of advection schemes [5], [6], [8].

In this article, we analyse the stability of some proposed hyperbolic schemes by using IDFTM which is much easier and faster than the conventional Fourier transform approach.

MATERIALS AND METHODS

In this section, some basic concepts and definitions that were used to prove our main results are provided.

Definition 1: The Amplification Factor $\lambda \equiv \frac{A\Delta t}{\Delta x}$ is the number derived using Von Neumann stability analysis to determine the stability of a numerical scheme for a partial differential equation. Where the necessary and sufficient condition for the scheme to be stable is $|\lambda| \leq 1$ [7].

Definition 2: A hyperbolic scheme is said to be Stable if it satisfies Von Neumann stability condition ($|\lambda| \leq 1$) and Unstable if and only if Von Neumann stability condition is not satisfied.
This implies that, for any scheme to be stable, when the obtained interval that satisfies Von Neumann intersects with the domain (0,1) and the result is (0,1), then it implies that the scheme is Conditionally Stable. If otherwise it is Unconditionally Stable [9].

Definition 3: The integrated decomposed Fourier transform of partial derivatives are defined as follows:

$$\partial U_i^n = U_{i+1}^n - U_{i-1}^n = 2U_i^n \quad (1)$$

$$\partial^2 U_i^n = U_{i+1}^n - 2U_i^n + U_{i-1}^n = -4U_i^n \quad (2)$$

$$\partial^2 U_i^{n-1} = U_{i+1}^{n-1} - 2U_i^{n-1} + U_{i-1}^{n-1} = -4U_i^{n-1} \quad (3)$$

Formulation of the Schemes

The proposed schemes were generated from making suitable substitutions and also combination of some of Forward and Backward Euler schemes and also, Lax-Fredrick's and Lax-Wendroff schemes as follows:

$$U_i^{n+1} + U_i^n = \frac{A\Delta t}{\Delta x}[U_{i+1}^n - U_{i-1}^n] \quad (4)$$

$$U_i^{n+1} = U_i^n + \frac{A\Delta t}{2\Delta x}[U_{i+1}^n - 2U_i^n + U_{i-1}^n] \quad (5)$$

$$U_i^{n+1} = U_i^n - \frac{A\Delta t}{\Delta x}[U_{i+1}^n - U_{i-1}^n] \quad (6)$$

$$U_i^{n+1} = U_i^n - 2\frac{A\Delta t}{\Delta x}[U_{i+1}^n - U_{i-1}^n] - \frac{K^2A^2}{2(\Delta x)^2}[U_{i+1}^n - 2U_i^n + U_{i-1}^n] \quad (7)$$

ANALYSIS OF THE RESULTS

Stability Analysis of Scheme (4) using IDFTM

From (4) we obtain:

$$U_i^{n+1} + U_i^n = \frac{A\Delta t}{\Delta x}[U_{i+1}^n - U_{i-1}^n] \quad (8)$$

and from (1) we have,

$$U_{i+1}^n - U_{i-1}^n = 2U_i^n$$

now (8) becomes

$$U_i^{n+1} = U_i^n - \frac{A\Delta t}{\Delta x}[2U_i^n]$$

Let $\frac{A\Delta t}{\Delta x} = \lambda$

$$\Rightarrow U_i^{n+1} = U_i^n - 2\lambda U_i^n$$

$$U_i^{n+1} = U_i^n[1 - 2\lambda]$$

$$\left|\frac{U_i^{n+1}}{U_i^n}\right| = |1 - 2\lambda| \leq 1$$

$$\Rightarrow -1 \leq 1 - 2\lambda \leq 1$$

$$0 \leq \lambda \leq 1$$

$$\Rightarrow \lambda \in (0,1)$$

It implies that if $\lambda \in (0,1)$, it is Conditionally Stable and also satisfies Von Neumann Convergence Conditions.

Stability Analysis of Scheme (5) using IDFTM

$$U_i^{n+1} = U_i^n + \frac{A\Delta t}{2\Delta x}[U_{i+1}^n - 2U_i^n + U_{i-1}^n] \quad (9)$$

From (2) we get,

$$\partial^2 U_i^n = U_{i+1}^n - 2U_i^n + U_{i-1}^n = -4U_i^n$$

Therefore (9) now becomes

$$U_i^{n+1} = U_i^n + \frac{A\Delta t}{2\Delta x}[-4U_i^n]$$

Let $\frac{A\Delta t}{\Delta x} = \lambda$

$$\Rightarrow U_i^{n+1} = U_i^n - 2\lambda U_i^n$$

$$U_i^{n+1} = U_i^n[1 - 2\lambda]$$

$$\left|\frac{U_i^{n+1}}{U_i^n}\right| = |1 - 2\lambda| \leq 1$$

$$\Rightarrow -1 \leq 1 - 2\lambda \leq 1$$

$$0 \leq \lambda \leq 1$$

$$\Rightarrow \lambda \in (0,1)$$

It implies that if $\lambda \in (0,1)$, it is Conditionally Stable and also satisfies Von Neumann Convergence Conditions.

Stability Analysis of Scheme (6) using IDFTM

$$U_i^{n+1} = U_i^n - \frac{A\Delta t}{\Delta x}[U_{i+1}^n - U_{i-1}^n] \qquad (10)$$

From (1) above we have,

$$U_{i+1}^n - U_{i-1}^n = 2U_i^n$$

Therefore, (10) now becomes

$$U_i^{n+1} = U_i^n - \frac{A\Delta t}{\Delta x}[2U_i^n]$$

Let $\frac{A\Delta t}{\Delta x} = \lambda$

$$\Rightarrow U_i^{n+1} = U_i^n - 2\lambda U_i^n$$

$$U_i^{n+1} = U_i^n[1 - 2\lambda]$$

$$\left|\frac{U_i^{n+1}}{U_i^n}\right| = |1 - 2\lambda| \leq 1$$

$$\Rightarrow -1 \leq 1 - 2\lambda \leq 1$$

$$0 \leq \lambda \leq 1$$

$$\Rightarrow \lambda \in (0,1)$$

It implies that if $\lambda \in (0,1)$, it is Conditionally Stable and also satisfies Von Neumann Convergence Conditions.

Stability Analysis of Scheme (7) using IDFTM

$$U_i^{n+1} = U_i^n - 2\frac{A\Delta t}{\Delta x}[U_{i+1}^n - U_{i-1}^n] - \frac{K^2A^2}{2(\Delta x)^2}[U_{i+1}^n - 2U_i^n + U_{i-1}^n] \qquad (11)$$

From (1) and (2) we have,

$$U_{i+1}^n - U_{i-1}^n = 2U_i^n$$

and

$$\partial^2 U_i^n = U_{i+1}^n - 2U_i^n + U_{i-1}^n = -4U_i^n$$

Therefore (11) now becomes:

$$U_i^{n+1} = U_i^n - 2\frac{A\Delta t}{\Delta x}[2U_i^n] - \frac{K^2A^2}{2(\Delta x)^2}[-4U_i^n]$$

Let $\frac{A\Delta t}{\Delta x} = \frac{K^2A^2}{(\Delta x)^2} = \lambda$

$$\Rightarrow U_i^{n+1} = U_i^n - 4\lambda U_i^n + 2\lambda U_i^n$$

$$U_i^{n+1} = U_i^n - 2\lambda U_i^n$$

$$U_i^{n+1} = U_i^n[1 - 2\lambda]$$

$$\left|\frac{U_i^{n+1}}{U_i^n}\right| = |1 - 2\lambda| \leq 1$$

$$\Rightarrow -1 \leq 1 - 2\lambda \leq 1$$

$$0 \leq \lambda \leq 1$$

$$\Rightarrow \lambda \in (0,1)$$

It implies that if $\lambda \in (0,1)$, it is Conditionally Stable and also satisfies Von Neumann Convergence Conditions.

APPLICATION OF THE METHOD

In this section, we shall apply the IDFTM to analyze Lax-Wendroff scheme and Staggered Leap-Frog scheme. This will show the stability of the schemes and the reliability of the proposed method.

Analysis of Lax-Wendroff Scheme

Given Lax-Wendroff scheme as:

$$U_i^{n+1} = U_i^n - \frac{KA}{2\Delta x}[U_{i+1}^n - U_{i-1}^n] + \frac{K^2A^2}{2(\Delta x)^2}[U_{i+1}^n - 2U_i^n + U_{i-1}^n] \qquad (12)$$

Substituting (1) and (2) into (12) we have,

$$U_i^{n+1} = U_i^n - \frac{\mathrm{KA}}{2\Delta \mathrm{x}}[2U_i^n] + \frac{\mathrm{K^2A^2}}{2(\Delta \mathrm{x})^2}[-4U_i^n]$$
$$= U_i^n - \frac{\mathrm{KA}}{\Delta \mathrm{x}}U_i^n - 2\frac{\mathrm{K^2A^2}}{(\Delta \mathrm{x})^2}U_i^n$$

Let

$$\frac{\mathrm{KA}}{\Delta \mathrm{x}} = \frac{\mathrm{K^2A^2}}{(\Delta \mathrm{x})^2} = \lambda$$

$$U_i^{n+1} = U_i^n - \lambda U_i^n - 2\lambda U_i^n$$

$$U_i^{n+1} = U_i^n[1 - 3\lambda]$$

$$\Rightarrow \left|\frac{U_i^{n+1}}{U_i^n}\right| = |1 - 3\lambda| \leq 1$$

$$\Rightarrow -1 \leq 1 - 3\lambda \leq 1$$

$$0 \leq \lambda \leq \frac{2}{3}$$

It implies that if $\lambda \in (0, \frac{2}{3})$, Lax-Wendroff scheme is Stable and also satisfies Von Neumann Convergence Conditions.

Analysis of Staggered Leap-Frog Scheme

Given Staggered Leap-Frog scheme as:

$$\left[\frac{U_i^{n+1} - U_i^{n-1}}{2\Delta \mathrm{t}}\right] = -A\left[\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta \mathrm{t}}\right] \quad (12)$$

$$\Rightarrow U_i^{n+1} = U_i^{n-1} - \frac{A\Delta \mathrm{t}}{\Delta \mathrm{x}}[U_{i+1}^n - U_{i-1}^n] \quad (13)$$

Substituting (1) and (2) into (13) we have,

$$U_i^{n+1} = U_i^{n-1} - 2\frac{A\Delta \mathrm{t}}{\Delta \mathrm{x}}U_i^n$$

Let $\frac{A\Delta \mathrm{t}}{\Delta \mathrm{x}} = \lambda$

$$\Rightarrow U_i^{n+1} = U_i^{n-1} - 2\lambda U_i^n \quad (14)$$

Let

$$V_i^{n+1} = U_i^n \quad (15)$$

$$\Rightarrow V_i^n = U_i^{n-1}$$

Now (14) and (15) becomes,

$$U_i^{n+1} = -2\lambda U_i^n + V_i^n \quad (16)$$

$$V_i^{n+1} = U_i^n \quad (17)$$

Therefore (16) and (17) now becomes:

$$\begin{pmatrix} U_i^{n+1} \\ V_i^{n+1} \end{pmatrix} = \begin{pmatrix} -2\lambda & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} U_i^n \\ V_i^n \end{pmatrix}$$

The characteristics equation is given as,

$$|A - \mu I| = 0 \quad (18)$$

$$\begin{vmatrix} -2\lambda - \mu & 1 \\ 1 & -\mu \end{vmatrix} = 0$$

$$\mu^2 + 2\lambda\mu - 1 = 0 \quad (19)$$

$$\mu = \frac{-2\lambda \pm \sqrt{4\lambda^2 + 4}}{2}$$

Let $\lambda^2 = 0$

Then

$$|\mu| = |-\lambda \pm 1| \leq 1$$

$$|\mu_1| = |-\lambda + 1|, \ |\mu_2| = |-\lambda - 1|$$

$$\Rightarrow \mu_1 : \lambda \in (0,2) \ and \ \ \mu_2 : \lambda \in (-2,0)$$

Since $\lambda \in \mu_1 \cap \mu_2$

$$\Rightarrow \lambda \in (0,2) \cap (-2,0)$$

$$\therefore \lambda \in (0,0)$$

Then it implies that Staggered Leap-Frog scheme is Stable and also satisfies Von Neumann Convergence Conditions.

CONCLUSION

This article has been able to analyze the proposed schemes by the substitution integrated decomposed Fourier transform method. Also, we have been able to show that the IDFTM is reliable and faster on applying it to Lax-Wendroff scheme and Staggered Leap-Frog scheme. From the results of the stability analysis using

IDFTM, we have also seen that the proposed schemes are all stable.

REFERENCES

1. Anderson, J.D., Jr. 1994. *Computational Fluid Dynamics: The Basics with applications.* McGraw Hill: New York, NY.

2. Charney, J.G., R. Fjortoft, and J. Von Neumann. 1950. "Numerical Integration of the Barotropic Velocity Equation". *Tellus.* 2:237-254.

3. Crank, J. and P. Nicolson. 1947. "A Practical Method for Numerical Evaluation of Solutions of Partial Differential Equations of Heat Conduction type". *Proceedings of the Cambridge Philosophical Society.* 43:50-67.

4. Felippa, C.A. 1994. "An Appreciation of R. Courants's Variational Methods for the Solution of Problems of Equilibrium and Vibrations". *International Journal for Numerical Methods in Engineering.* 37:2159-2187.

5. Jovanovic, B.J. and E. Suli. 2014. "Analysis of Finite Difference Schemes". *Springer Series in Computational Mathematics*. 94(11).

6. Kent, J., C. Jablonowski, and R. Rood. 2014. "Determining the effective resolution of advection schemes Part II: Numerical testing". *Journal of Computational Physics.* 278:497-508.

7. Smith, G.D. 1985. "Numerical Solution of Partial Differential Equations". *Finite Difference Methods.* 3:67-68.

8. Tadmor, E. 2012. "A review of numerical methods for nonlinear partial differential equations". *Bulletin of the American Mathematical Society*. 49(4):507-554.

9. Zienkicicz, O.C. and K. Morgan. 1983. *Finite Element and Approximation.* John Wiley and Sons, Inc.: New York, NY.

ABOUT THE AUTHORS

Aniki Samuel Adamariko holds a B.Sc. (Hons) degree in Mathematics from University of Ilorin, Nigeria and an M.Tech. in Mathematics from the Federal University of Technology, Minna, Nigeria. Aniki Samuel Adamariko is in the process of obtaining a Ph.D. in Mathematics at University of Ilorin, Nigeria.

Dr. Kamilu Rauf has obtained his Ph.D. degree in Functional Analysis. He has authored several articles. In 2010, he was awarded a prestigious Scheme 5 Fellowship by the London Mathematical Society and presented lectures at several universities in the United Kingdom.

SUGGESTED CITATION

Aniki, S.A. and K. Rauf. 2015. "Integrated Decomposed Fourier Transform Method for Convergence Analysis of Some Hyperbolic Schemes". *Pacific Journal of Science and Technology*. 10(2):98-102.

Pacific Journal of Science and Technology