Finite Element Method on Derivative Least-Square and Semi-Standard Galerkin for Solving Boundary Value Problems.

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ABSTRACT

In this paper, we derived **a** Derivative Least–Square Method (DLSM) and Semi-Standard Galerkin Method (SSGM) which are both Finite Element Methods (FEM) for Solving Boundary Value Problems. In the DLSM, the residue was differentiated into a new derivative of the residue, $R' = \frac{d}{dx}(Lu - f)$ while in the SSGM, we took $I = \int_a^b Du_i R(x, c_i)\, dx = 0; \quad i = 1, 2, \ldots, n$, for which $Du_i(x)$ is its basis function. The proposed methods were applied on several boundary value problems and the numerical results obtained are reliable and in good agreement with the exact solutions.

(Keywords: boundary value problem, derivative least-square method, exact solution, finite element method, residue, semi-standard Galerkin method)

INTRODUCTION

This paper was motivated by the successful application of FEM to the solution of Ordinary Differential Equation. But numerical problems occurring in fluid and transport problems showed that the Galerkin methods could exhibit numerical problems, if it is applied to non- self adjoint partial differential equations [4]. As a consequence, first works which casted arbitrary partial differential equations into an equivalent minimization problem using a least squares principle in conjunction with the finite element ideas appeared also by that time [11], [15]. A huge part of the theoretical analysis of least squares methods is connected to the theory of elliptic systems.

A major work in this area is a series of two papers from Agmon, Douglas and Nirenberg, who developed the ADN-Theory which is used in many later publications on the least squares finite element method (LSFEM) [1]. Later on Wendland used tools from complex function theory to establish several theorems for elliptic systems in two dimensional domains [13]. He established that application of a least squares principle may lead to suboptimal convergence. He points out that appropriate weight must be introduced to obtain optimal accuracy. After that a more general theory especially for the LSFEM was developed by Aziz [3]. They utilized the ADN-Theory to get a priori estimates for elliptic systems which then allow proving optimal convergence rates (with respect to the used elements).

Beside these general theoretical works, several results have been published which considered special problems and their treatment with the LSFEM. Recently, Bortolot and Karam-Filho worked on a stabilized finite element analysis for a power law pseudo-plastic stokes problem. Also, Lee and Chen published Adaptive Least-Squares finite element approximations to stokes equations, Arnold and Awanou on Finite element differential forms on cubical meshes [2], [9].

The area of numerical analysis comprises of several methods, Gelarkin methods are classes of methods for converting a differential equation to a discrete problems. It is in principle the same as applying the method of variation of parameters to a function space, by converting the equation to a weak formulation. When referring to a Galerkin method, we also need to give the name along with typical approximation methods used, such as Bubnov-Galerkin method, Petrov-Galerkin method or Ritz-Galerkin method [7]. In most mathematical modelling problems differential equations of specific forms are derived which describe a system. Values of the coefficients, which can be constants or functions, of the differential equations are usually specified, and the solutions are calculated or presented in closed form, with little, if any, indication of how the coefficients can be estimated from observations.

In subsequent times several methods of solving Boundary Value Problems using finite element Galerkin's methods has been used [5], [8], [10], [16]. However, recently Jin, Lazarow Liu and Zhou worked on The Galerkin finite element method for a multi-term time fractional diffusion equation, Harris and Harris on Supper convergence of weak Galerkin finite element approximation for second order elliptic problems by L_2 –projections, Wong and Candes on Numerically stable finite element methods for the Galerkin solution of eddy current problems [6], [7], [14]. In 2015, Rauf et al [11] used linear multistep method toobtain a zero-stable block method for the solution of third order ordinary differential equations.

MATERIALS AND METHODS

General Form of the Residue

Let the boundary value problem with the boundary condition $y(a) = A;\ y(a) = A$ be
$y'' + p(x)y' + q(x)y = f(x);\ \ \forall x \epsilon [a, b]$

Let the approximate solution be :

$$u(x) = u_0 + \sum_{i=1}^{n} c_i u_i$$

$$L[u] = L[u_0] + [\sum_{i=1}^{r} c_i u_i] = f$$

$$R = (L[u^0] - f) + L[\sum_{i=1}^{r} c_i u_i] \qquad (1)$$

where R is the Residue.

Derivation of DLSM

From (1), we let
$R = Lu - f$
Giving the Least-Square method

$$I = min \int_a^b (Lu - f)^2 dx;\ \ x\epsilon[a, b]$$

$$I = min \int_a^b R^2 dx \qquad (2)$$

Replacing R with $\frac{dR}{dx}$ in (2), leads to our main result:

$$I = min \int_a^b [\frac{d(Lu - f)}{dx}]^2 dx$$

$$I = \int_a^b (\frac{dR}{dx})^2 dx$$

Let $\frac{dR}{dx} = R'$

$$I = \int_a^b (R')^2 dx;\ \ x\epsilon[a, b] \qquad (3)$$

where $\frac{\partial I}{\partial c_i} = 0\ ;\ \ i = (1,2, \dots , n) \qquad (4)$

by solving the n – system of Equation (4) and also making use of (3) to obtain the values of c_i. The approximate numerical solution is now giving as follows:

$$u(x) = u_0(x) + \sum_{1}^{n} c_i u_i\ ,\ \forall\ i = (1,2, \dots , n)$$

where

$$u_0(x) \Rightarrow y(x) = y_1 + \frac{(y_2 - y_1)(x - x_1)}{(x_2 - x_1)}$$

for the boundary value conditions

$y(a) = B\ \equiv y(x_1) = A$ and
$y(b) = B\ \equiv y(x_2) = B$

Derivation of SSGM

Also from (1), Let the basis function be v_i;
It implies that:

$$(Lu - f, v_i) = 0$$
$$(Lu, v_i) - (f, v_i) = 0$$
$$(Lu, v_i) = (f, v_i) \qquad (5)$$

Now making use of (2);
For Standard Galerkin (H^0 - Galerkin)
$D^0(Lu, v_i) = D^0(f, v_i)$

$(D^0 Lu, D^0 v) = (D^0 f, D^0 v_i)$
$(Lu, D^0 v_i) = (f, D^0 v_i)$
$(Lu, v_i) = (f, v_i)$

Which is the standard Galerkin method. This implies that:

$$D^{\frac{n}{2}}(Lu, v_i) = D^{\frac{n}{2}}(f, v_i)$$
$$\left(D^{\frac{n}{2}}Lu, D^{\frac{n}{2}}v_i\right) = (D^{\frac{n}{2}}f, D^{\frac{n}{2}}v_i)$$
$$(Lu, D^n v_i) = (f, D^n v_i) \qquad (6)$$

putting n = 1 gives us the Proposed Semi – Standard Galerkin $\left(H^{\frac{1}{2}} - Gealerkin\right)$ which is expressed as follows:

$$D^{\frac{1}{2}}(Lu, v_i) = D^{\frac{1}{2}}(f, v_i)$$
$$\left(D^{\frac{1}{2}}Lu, D^{\frac{1}{2}}v_i\right) = (D^{\frac{1}{2}}f, D^{\frac{1}{2}}v_i)$$
$$(Lu, Dv_i) = (f, Dv_i) \qquad (7)$$

Therefore, for the basis:

$$u_i = (x - x^{i+1})$$
$$Du_i = [1 - (i+1)x^i] \quad , \forall i = (1,2,\dots,n) \qquad (8)$$

and for $x\epsilon[a, b]$,

$$I = \int_a^b Du_i R(x, c_i)\, dx = 0; \quad i = (1,2,\dots,n) \qquad (9)$$

where R is the residue of the boundary value problem. On solving the n – system of Equation (9) and by making use of (8) to obtain the values of c_i. The approximate numerical solution is now giving as follows:

$$u(x) = u_0(x) + \sum_1^n c_i u_i \, , \forall i = (1,2,\dots,n) \qquad (10)$$

and

$$u_0(x) \Rightarrow y(x) = y_1 + \frac{(y_2 - y_1)(x - x_1)}{(x_2 - x_1)} \qquad (11)$$

for the boundary value conditions:

$y(a) = B \equiv y(x_1) = A$ and
$y(b) = B \equiv y(x_2) = B$

APPLICATION OF THE METHODS AND RESULTS

To test the applicability of the proposed methods, we consider the following boundary value problems;

Problem 1: Consider the following boundary value problem with its boundary condition

$$y'' - y' = -2x; \; y(0) = 0, \qquad y(1) = 3$$

Problem 2: Consider the following boundary value problem with its boundary condition

$$y'' - 2y = 0; \; y(0) = 1, \; y(\pi) = 0,$$

whose exact solution is

$$y(x) = cos\sqrt{2}x - cot\sqrt{2}\pi . sin\sqrt{2}x$$

The results obtained are tabulated in Tables 1 and 2.

Solution of Problem 1 using Least-Square Method

Let

$$u(x) = u_0(x) + c_1 u_1(x) + c_2 u_2(x) \qquad (12)$$

where

$$u_0(x) = y_1 + \frac{(y_2 - y_1)(x - x_1)}{(x_2 - x_1)} = 3x$$
$$u_1(x) = x - x^2 \; and \; u_2(x) = x - x^3$$
$$\Rightarrow u(x) = 3x + c_1(x - x^2) + c_2(x - x^3) \qquad (13)$$

$$u'(x) = 3 + c_1(1 - 2x) + c_2(2x - 3x^2)$$
$$u''(x) = -2c_1 + c_2(2 - 6x)$$
$$u''(x) - u'(x) = -2c_1 + c_2(2 - 6x) - 3 - c_1(1 - 2x) - c_2(2x - 3x^2) = -2x$$
$$R = c_1(2x - 3) + c_2(3x^2 - 6x + 1) + 2x - 3 \qquad (14)$$

Since the method is given by:

$$I = \int_a^b R^2 dx; \quad x \epsilon [a, b]$$

Therefore,

$$I = \int_0^1 [c_1(2x-3) + c_2(3x^2 - 6x + 1) + 2x - 3]^2$$

$$= \int_0^1 c_1^2 (2x-3)^2 dx + \int_0^1 c_2^2 (3x^2 - 6x + 1)^2 \, dx + \int_0^1 (2x-3)^2 dx + 2c_1 c_2 \int_0^1 (2x - 3)(3x^2 - 6x + 1) dx + 2c_1 \int_0^1 (2x-3)^2 + 2c_2 \int_0^1 (2x-3)(3x^2 - 6x + 1) dx$$

$$= c_1^2 \int_0^1 (4x^2 - 12x + 9) dx + c_2^2 \int_0^1 (9x^4 - 36x^3 - 30x^2 - 12x + 1) dx + \int_0^1 (4x^2 - 12x + 9) dx + 2c_1 c_2 \int_0^1 (6x^3 + 21x^2 + 20x - 3) dx + 2c_1 \int_0^1 (4x^2 - 12x + 9) dx + 2c_2 \int_0^1 (9x^4 - 36x^3 - 30x^2 - 12x + 1) dx$$

By taking the limit we obtain the following:

$$= c_1^2 \left({}^4\!/_3 - 6 + 9 \right) + c_2^2 \left({}^2\!/_5 + 12 - {}^9\!/_2 - 2 + 6 \right) + \left({}^4\!/_3 - 6 + 9 \right) + 2c_1 c_2 \left({}^3\!/_2 - 7 + 8 + 3 \right) + 2c_1 \left({}^4\!/_3 - 6 + 9 \right) + 2c_2 \left({}^3\!/_2 - 7 + 8 + 3 \right)$$

$$I = \frac{13}{3} c_1^2 + \frac{133}{10} c_2^2 + \frac{13}{3} + 11 c_1 c_2 + \frac{26}{3} + 11 c_2$$

$$\frac{\partial I}{\partial c_1} = \frac{26}{3} c_1 + 11 c_2 + \frac{26}{3} = 0$$

$$26c_1 + 33c_2 = -26 \qquad (15)$$

$$\frac{\partial I}{\partial c_2} = 11c_1 + \frac{133}{5} c_2 + 11 = 0$$

$$55c_1 + 133c_2 = -55 \qquad (16)$$

On solving (15) and (16) simultaneously, we obtain:

$c_1 = -1,$
$c_2 = 0$

On substituting the values of c_1 and c_2 into (12) we have:

$$y = u(x) = 2x + x^2 \qquad (17)$$

From (17)

when $x = 0, \ y = 0$
when $x = 0.5, \ y = 1.25$
and
when $x = 1, \ y = 3$

Solution of Problem 1 using Standard Galerkin Method

Let

$$u(x) = u_0(x) + c_1 u_1(x) + c_2 u_2(x) \qquad (18)$$

where

$$u_0(x) = y_1 + \frac{(y_2 - y_1)(x - x_1)}{(x_2 - x_1)} = 3x$$

$u_1(x) = x - x^2$ *and* $u_2(x) = x^2 - x^3$

$$\Rightarrow u(x) = 3x + c_1(x - x^2) + c_2(x^2 - x^3) \qquad (19)$$

$$u'(x) = 3 + c_1(1 - 2x) + c_2(2x - 3x^2)$$
$$u''(x) = -2c_1 + c_2(2 - 6x)$$
$$u''(x) - u'(x) = -2c_1 + c_2(2 - 6x) - 3 - c_1(1 - 2x) - c_2(2x - 3x^2) = -2x$$
$$R = c_1(2x - 3) + c_2(3x^2 - 8x + 2) + 2x - 3 \qquad (20)$$

$$I = \int_a^b u_i R(x, c_i) \, dx = 0; \quad i = 1,2$$

Firstly, for $u_1(x) = x - x^2$ we have:

$$\int_0^1 (x - x^2)[c_1(2x - 3) + c_2(3x^2 - 8x + 2)] dx = \int_0^1 (2x - 3)(x - x^2) dx \qquad (21)$$

$$c_1\int_0^1(2x^2-3x-2x^3+3x^2)dx + c_2\int_0^1(3x^3-8x^2+2x-3x^4 + 8x^3-2x^2)\,dx = \int_0^1(3x-3x^2-2x^2+2x^3)dx$$

By taking the limit we have:

$$c_1\left(\frac{2}{3}-\frac{3}{2}-\frac{1}{2}+\frac{2}{4}\right)+c_2\left(\frac{3}{4}-\frac{8}{3}+1-\frac{3}{5}+2-\frac{2}{3}\right) = \left(\frac{3}{2}-1-\frac{2}{3}+\frac{1}{2}\right)$$

$$\Rightarrow -\frac{c_1}{3}-\frac{11}{60}c_2=\frac{1}{3}$$

$$-20c_1-11c_2 = 20 \qquad (22)$$

Secondly, for $u_1(x)=x^2-x^3$ we have:

$$\int_0^1(x^2-x^3)[c_1(2x-3)+c_2(3x^2-8x+2)]dx = \int_0^1(2x-3)(x^2-x^3)dx \qquad (23)$$

$$c_1\int_0^1(2x^3-3x^2-2x^4+3x^3)dx + c_2\int_0^1(3x^4-8x^3+2x^2-3x^5 + 8x^4-2x^3)\,dx = \int_0^1(3x^2-3x^3-2x^3+2x^4)dx$$

By taking the limit we have:

$$c_1\left(\frac{1}{2}-1-\frac{2}{5}+\frac{3}{4}\right)+c_2\left(\frac{3}{5}-2+\frac{2}{3}-\frac{1}{2}+\frac{8}{5}-\frac{1}{2}\right) = \left(1-\frac{3}{4}-\frac{1}{2}+\frac{2}{5}\right)$$

$$\Rightarrow -\frac{3}{20}c_1-\frac{2}{15}c_2=\frac{3}{20}$$

$$-9c_1-8c_2 = 9 \qquad (24)$$

On solving (22) and (24) simultaneously, we obtain:

$$c_1=-1,\quad c_2 = 0 \qquad (25)$$

By substituting the values of c_1 and c_2 into (19) we have:

$$y=u(x) = 2x+x^2 \qquad (26)$$

From (26)

when $x=0,\ y=0$
when $x=0.5,\ y=1.25$
and
when $x=1,\ y=3$

Solution of Problem 1 using DLSM

Let

$$u(x)=u_0(x)+c_1u_1(x) + c_2u_2(x) \qquad (27)$$

where

$$u_0(x)=3x$$
$$u_1(x)=x-x^2 \text{ and } u_2(x)=x^2-x^3$$
$$\Rightarrow u(x)=3x+c_1(x-x^2) + c_2(x^2-x^3) \qquad (28)$$

$$u'(x)=3+c_1(1-2x)+c_2(2x-3x^2)$$
$$u''(x)=-2c_1+c_2(2-6x)$$
$$u''(x)-u'(x)=-2c_1+c_2(2-6x)-3 - c_1(1-2x)-c_2(2x-3x^2) = -2x$$
$$R=c_1(2x-3)+c_2(3x^2-8x+2)+2x-3$$
$$R'=2c_1+c_2(6x-8) + 2 \qquad (29)$$

The method is given by:

$$I=\int_a^b(R')^2dx;\quad x\epsilon[a,b]$$

Therefore,

$$I=\int_0^1[(2c_1+2)+c_2(6x-8)]^2 = \int_0^1(2c_1+2)^2\,dx + \int_0^1(2c_1+2)(6c_2x-8c_2)\,dx + \int_0^1(6c_2x-8c_2)^2\,dx$$

By taking the limit we obtain the following:

$$I = 8c_1 - 10c_2 - 10c_1c_2 + 4c_1^2 + 28c_2^2 + 4$$
$$\frac{\partial I}{\partial c_1} = 8c_1 - 10c_2 + 8 = 0$$
$$8c_1 - 10c_2 = -8 \qquad (30)$$

$$\frac{\partial I}{\partial c_1} = -10c_1 + 56c_2 - 10 = 0$$
$$-10c_1 + 56c_2 = 10 \qquad (31)$$

By solving (30) and (31) simultaneously we obtain:

$$c_1 = -1, \quad c_2 = 0$$

By substituting the values of c_1 and c_2 into (28) we have:

$$y = u(x) = 2x + x^2 \qquad (32)$$

From (32)

when $x = 0,\ y = 0$
when $x = 0.5,\ y = 1.25$
and
when $x = 1,\ y = 3$

Solution of Problem 1 using SSGM

Let

$$u(x) = u_0(x) + c_1u_1(x) + c_2u_2(x) \qquad (33)$$

where

$$u_0(x) = 3x$$
$$u_1(x) = 1 - x \ \ and \ \ u_2(x) = x - x^2$$
$$Du_1(x) = -1 \ \ and \ \ Du_2(x) = 1 - 2x$$
$$\Rightarrow u(x) = 3x + c_1(1 - x) + c_2(x - x^2) \qquad (34)$$

$$u'(x) = 3 - c_1 + c_2(1 - 2x)$$
$$u''(x) = -2c_2$$
$$R = u''(x) - u'(x) = c_1 + c_2(2x - 3) = 3 - 2x \qquad (35)$$

Since,

$$I = \int_a^b Du_iR(x, c_i)\,dx = 0; \quad i = 1,2$$

Firstly, for $Du_1(x) = -1$ we have

$$\int_0^1 [c_1 + c_2(2x - 3)]dx = \int_0^1 (3 - 2x)\,dx \qquad (36)$$

By taking the limit we have:

$$c_1 - 2c_2 = 2 \qquad (37)$$

Secondly, for $Du_2(x) = 1 - 2x$ we have:

$$\int_0^1 (1 - 2x)[c_1 + c_2(2x - 3)]dx = \int_0^1 (1 - 2x)(3 - 2x)\,dx \qquad (38)$$

By taking the limit we have:

$$c_1(1 - 1) + c_2\left(4 - 3 - \frac{4}{3}\right) = \left(3 - 4 + \frac{4}{3}\right)$$
$$\Rightarrow c_2 = -1 \qquad (39)$$

By solving (37) and (39) we obtain:

$$c_1 = 0, \quad c_2 = -1 \qquad (40)$$

By substituting the values of c_1 and c_2 into (34) we have:

$$y = u(x) = 2x + x^2 \qquad (41)$$

From (41)

when $x = 0,\ y = 0$
when $x = 0.5,\ y = 1.25$
and
when $x = 1,\ y = 3$

The tables below shows the results obtained from computing the test problems and comparing the results of the proposed methods with their exact solution and solution of some existing methods.

Table 1: Comparing the Value of the Proposed Method with some Exiting Methods for Problem 1.

X	Approximate Value of Least-Square Method	Approximate Value of Galerkin Method	Approximate Value of SSGM	Approximate Value of DLSM
0.0	0.00	0.00	0.00	0.00
0.1	$2.1E-1$	$2.1E-1$	$2.1E-1$	$2.1E-1$
0.2	$4.4E-1$	$4.4E-1$	$4.4E-1$	$4.4E-1$
0.3	$6.9E-1$	$6.9E-1$	$6.9E-1$	$6.9E-1$
0.4	$9.6E-1$	$9.6E-1$	$9.6E-1$	$9.6E-1$
0.5	1.25	1.25	1.25	1.25
0.6	1.56	1.56	1.56	1.56
0.7	1.89	1.89	1.89	1.89
0.8	2.24	2.24	2.24	2.24
0.9	2.61	2.61	2.61	2.61
1.0	3.00	3.00	3.00	3.00

Table 2: Comparing the Value of the Proposed Method with the Exact Value for Problem 2.

X	Exact Value	Approximate Value of SSGM	Approximate Value of DLSM	Error of SSGM	Error of DLSM
0.1	0.999315153	0.998365914	0.998843180	$2.6201962E-4$	$4.719730000E-4$
0.2	0.998624219	0.997623840	0.998000350	$1.0003790E-3$	$6.238690543E-4$
0.3	0.997929200	0.996823832	0.996810654	$1.1033680E-3$	$1.116546531E-3$
0.4	0.997224102	0.996103019	0.996025610	$1.1210832E-3$	$1.198492000E-3$
0.5	0.996514928	0.995163768	0.995154709	$1.3511598E-3$	$1.360219000E-3$
0.6	0.995799683	0.994279498	0.994206001	$1.5201841E-3$	$1.593682000E-3$
0.7	0.995078372	0.993379936	0.993216358	$1.6984360E-3$	$1.862014000E-3$
0.8	0.994350998	0.992540752	0.992379866	$1.8102459E-3$	$1.971132000E-3$
0.9	0.993617566	0.991689155	0.991474806	$1.928411E-3$	$2.142760000E-3$
1.0	0.992878081	0.990594171	0.990493361	$2.283910E-3$	$2.384720000E-3$

CONCLUSION

The derived DLSM and SSGM of Finite Element Method are applied to solve Boundary Value Problems to test the efficiency of the methods. The numerical results obtained by the proposed methods as seen from Table 1 and Table 2 are reliable and in good agreement with their exact values and approximate solutions of the other methods.

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