Using Modified Euler Method (MEM) for the Solution of some First Order Differential Equations with Initial Value Problems (IVPs).

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ABSTRACT

Parker and Sochacki showed that a large class of ODE's could be converted to polynomial form using substitutions and using a system of equation. While this class of ODE's is dense in the analytic functions, it does not include all analytic functions. They also showed one can approximate the solution by a polynomial system and the resulting error bound when using these approximation. Differential equations can be solved using many methods that are generally accepted in mathematics. However, it is believed that one method is more accurate, efficient, sufficient and unique than the other. Thus; solutions of First Order Differential Equations (FODEs) with Initial Value Problems (IVPs) by Modified Euler Methods (MEM) will be exercised. Explicitly, numerical computational algorithm, convergence rate, approximation errors and uniqueness is of sensitive attention and concern.

(Keywords: Modified Euler Method, MEM, Exact Solutions, ES, Initial Value Problem, IVP, convergence rate, analytical solution, numerical solution, error estimate, First Order Differential Equation, FODE)

INTRODUCTION

Parker and Sochacki theorem on Existence and Uniqueness states that if both $f(x, y)$ and $\frac{\partial y}{\partial x}$ are continuous in some region around the point (x_0, y_0) then there is a unique solution to the IVP [1]:

$$\begin{cases} y' & = f(x, y) \\ y(x_0) & = y_0 \quad (IVP) \end{cases} \qquad (1)$$

Valid in some interval around x_0. In other words, if the slope field is sufficiently smooth at each point, then there is unique integral curve passing through any given point. How do we prove such a theorem? This method uses a sequence of approximate solutions and prove that these approximations converge at least in a small interval around x_0.

Euler Method is quite simple to use in practice: one simply ''connect the dots'' in the slope field. The disadvantage to this method is that it only gives an approximation ''at the dots''. In other words, Euler Method only approximates the values of the solution at a finite list of points. It does not give us formula for an approximate function at every point. However, Euler Method has the advantage that its accuracy can be improved with only minor modifications.

Foremost applications some version of an improved Euler method is ideal. Eulerian methods, is far more efficient computationally than other methods such as Picard method but it introduces an important technique that will be useful for the error analysis of Eulerian methods. An approximation method is useless without an estimate of the error.

Material and Method

Modified Euler Method (MEM)

Considering the FODE with the IVP in Equation (1), then the solution to Equation (1) is equivalently defined by Equation (2):

$$\left.\begin{array}{c} \Rightarrow y_{i+1}^{(0)} = y_i + hf(x_i, y_i) \\ ie\ y_{((i+1),n)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f\left(x_{i+1}, y_{((i+1),n)}\right)\right] \\ f0r\ i = n = 0,1,2,3, \ldots \end{array}\right\} \qquad (2)$$

Proof:

To show that Equation (2) is the general solution to Equation (1) for Modified Euler Method (MEM) then: Let $y' = f(x, y)$ be the FODE with IVP $y(x_0) = y_0$ *ie from* equation (1)

$$\Rightarrow \frac{dy}{dx} = f(x, y) \Rightarrow dy = f(x, y)\ dx = \int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\ dx$$

$$\Rightarrow [y]_{y_0}^{y} = \int_{x_0}^{x} f(x, y)\ dx = y - y_0 = \int_{x_0}^{x} f(x, y)\ dx$$

$$\Rightarrow y_1 = y_1 + \int_{x_0}^{x} f(x, y)\ dx \qquad (3)$$

The integration of the RHS can be done using any numerical method. If the trapezoidal rule is used with step size $h(= x_1 - x_0)$ then the above integration becomes:

$$y(x) = y(x_0) + \frac{h}{2}\left[f\left(x_0, y(x_0)\right) + f\left(x_1, y(x_1)\right)\right] \qquad (4)$$

REMARK:

The RHS of Equation (4) involves an unknown quantity $y(x_1)$. This value can be determined by the Euler Method (EM). Denoting this value by $y^0(x_1)$ and is obtained through Equation (6) so as to completely give the $y^{(1)}(x_1)$. Then the resulting formula for finding y_1 becomes:

$$y_1(x_1) = y(x_0) + \frac{h}{2}\left[f\left(x_0, y(x_0)\right) + f\left(x_1, y^0(x_1)\right)\right] \qquad (5)$$

$$\Rightarrow y_1^{(0)} = y_0 + hf(x_0, y_0) \qquad (6)$$

$$ie\ y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^0)\right] \qquad (7)$$

Equation (7) gives the first approximation of y_1. The second approximation is:

$$ie\ y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^1)\right] \qquad (8)$$

The $(k+1)th$ approximate value of y_1 is:

$$ie\ y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^k\right)\right] \quad (9)$$

Generally;

$$\left.\begin{array}{l} \Rightarrow y_{i+1}^{(0)} = y_i + hf(x_i, y_i) \\ ie\ y_{((i+1),n)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f\left(x_{i+1}, y_{((i+1),n)}\right)\right] \\ f0r\ i = n = 0,1,2,3,\ldots \end{array}\right\} \quad (10)$$

The iterations are continued until two successive approximations $y_{i+1}^{(n)}$ and $y_{i+1}^{(n+1)}$ coincide to the desired accuracy. The iterations converge rapidly for sufficiently small spacing value of h.

Solution of Problem Using Modified Euler Method (MEM)

In this method, problem of the form in Equation (1) will be solved using Modified Euler Method.

Problem 3

Find the values of $y(0.1)$ and $y(0.2)$ from the following differential equation
$\frac{dy}{dx} = x^2 + y$
with initial condition
$y(0) = 0$. Also find the values of $y(0.1)$ and $y(0.2)$

Solution 3

From the given general term of Modified Euler Method in Equations ((9) and (10)):

$$\text{ie } y_{((i+1),n)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f\left(x_{i+1}, y_{((i+1),n)}\right)\right] \text{f0r } i = n = 1,2,3,\ldots$$

$$y_{((i+1),n)} = y_i + hf(x_i, y_i), \text{for } i = n = 0$$

Using a desired step size of h= 0.05

By the IVP:

$y(0) \Rightarrow$ value of y at $x = 0$

ie $y = 0$ when $x = 0$ (Initial Condition (IC)),

where $f(x,y) = x^2 + y$ by equation (1)

at $x_0 = 0, y_0 = 0, h = 0.05, i = n = 0$, and $f(x_0, y_0) = x_0^2 + y_0$

$$y_{((1),0)} = y_0 + hf(x_0, y_0)$$

$$y_{((1),0)} = y_0 + h(x_0^2 + y_0)$$
$$= 0 + (0.05)[(0)^2 + 0]$$

hence $y_{((1),0)} = 0.00000 = 0$

ie: $y_{((1),0)} = 0$ (11)

at $x_0 = 0, x_1 = x_0 + h = 0 + 0.05 = 0.05, y_0 = 0, h = 0.05, i = 0 =, n = 1,$

$f\left(x_1, y_{((1),0)}\right) = x_1^2 + y_{((1),0)}, y_{((1),0)} = 0$and $f(x_0, y_0) = x_0^2 + y_0$

$$y_{((0+1),1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_{0+1}, y_{((0+1),0)}\right)\right]$$

$$y_{((1),1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_{((1),0)}\right)\right]$$
$$= y_0 + \frac{h}{2}\left[(x_0^2 + y_0) + \left(x_1^2 + y_{((1),0)}\right)\right]$$

$$y_{((1),1)} = 0 + \left(\frac{0.05}{2}\right)[(0) + ((0.05)^2 + 0)]$$
$$= (0.025)[(0.0025)] = 0.0000625$$

ie $y_{((1),1)} = 0.0000625$

hence $y_{((1),1)} \cong 0.0001$ (12)

at $x_0 = 0, y_0 = 0, h = 0.05, i = 0, n = 2,$ and $f(x_0, y_0) = x_0^2 + y_0$

$f\left(x_1, y_{((1),2)}\right) = x_1^2 + y_{((1),1)}, y_{((1),1)} = 0.0000625, f(x_0, y_0) = x_0^2 + y_0 = 0$

$x_2 = x_1 + h = 0.05 + 0.05 = 0.1.$ie $x_1 = 0.05$ and $x_2 = 0.1$

$$y_{((0+1),2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_{((1),1)}\right)\right]$$

$$y_{((1),2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_{((1),1)}\right)\right]$$
$$= y_0 + \frac{h}{2}\left[(x_0^2 + y_0) + \left(x_1^2 + y_{((1),1)}\right)\right]$$

$$y_{((1),2)} = 0 + \left(\frac{0.05}{2}\right)[(0) + ((0.05)^2 + 0.0000625)]$$
$$= (0.025)[(0.0025625)] = 6.40625 \times 10^{-5}$$

ie $y_{((1),2)} = 6.40625 \times 10 -^{-5}$

hence $y_{((1),2)} \cong 0.0001$ (13)

at $x_0 = 0, y_0 = 0, h = 0.05, i = 1, n = 0$, and $f\left(x_1, y_{((1),2)}\right) = x_1^2 + y_{((1),2)}$

$y_{((1),2)} = 6.40625 \times 10 -^{-5}$

$x_1 = x_0 + h = 0 + 0.05 = 0.05.\text{ie } x_1 = 0.05$

$$y_{((2),0)} = y_{((1),2)} + hf\left(x_1, y_{((1),2)}\right)$$

$$\begin{aligned} y_{((2),0)} &= y_{((1),2)} + h\left(x_1^2 + y_{((1),2)}\right) \\ &= 6.40625 \times 10^{-5} + (0.05)[(0.05)^2 + 6.40625 \times 10^{-5}] \\ &= 6.40625 \times 10^{-5} + (0.05)[(2.5640625 \times 10^{-3})] \\ &= 6.40625 \times 10^{-5} + (1.28203125 \times 10^{-4}) \\ &= 1.92265625 \times 10^{-4} \end{aligned}$$

thus; $y_{((2),0)} \cong 0.0002$ (14)

at $x_0 = 0, y_0 = 0, h = 0.05, i = n = 1, f\left(x_1, y_{((1),2)}\right) = x_1^2 + y_{((1),2)}$,

$f\left(x_2, y_{((2),0)}\right) = x_2^2 + y_{((2),0)}, x_1^2 + y_{((1),2)} = 2.5640625 \times 10^{-3}$

$, y_{((2),0)} = 1.92265625 \times 10^{-4}, x_2 = x_1 + h = 0.05 + 0.05 = 0.1.$

ie : $x_1 = 0.05$, $x_2 = 0.1$ and $y_{((1),2)} = 6.40625 \times 10^{-5}$.

$$y_{((2),1)} = y_{((1),2)} + \frac{h}{2}\left[f\left(x_1, y_{((1),2)}\right) + f\left(x_2, y_{((2),0)}\right)\right]$$

$$y_{((2),1)} = y_{((1),2)} + \frac{h}{2}\left[\left(x_1^2 + y_{((1),2)}\right) + \left(x_2^2 + y_{((2),0)}\right)\right]$$

$$\begin{aligned} y_{((2),1)} &= 6.40625 \times 10^{-5} \\ &\quad + \left(\frac{0.05}{2}\right)[((0.05)^2 + 6.40625 \times 10^{-5}) + ((0.1)^2 + 1.92265625 \times 10^{-4})] \end{aligned}$$

$$\begin{aligned} &= 6.40625 \times 10^{-5} \\ &\quad + (0.025)[(0.0025 + 6.40625 \times 10^{-5}) + (0.01 + 1.92265625 \times 10^{-4})] \end{aligned}$$

$$= 6.40625 \times 10^{-5} + (0.025)(1.275632813 \times 10^{-2})$$

$$= 6.40625 \times 10^{-5} + 3.189082031 \times 10^{-4}$$

ie $y_{((2),1)} = 3.829707031 \times 10^{-4}$

$$\text{thus; } y_{((2),1)} \cong 0.0004 \tag{15}$$

$\text{at } h = 0.05, i = 1, n = 2, f\left(x_1, y_{((1),2)}\right) = x_1^2 + y_{((1),2)},$

$f\left(x_2, y_{((2),1)}\right) = x_2^2 + y_{((2),1)}, x_1^2 + y_{((1),2)} = 2.5640625 \times 10^{-3}$

$, y_{((2),1)} = 3.829707031 \times 10 -^{-4}, x_2 = x_1 + h = 0.05 + 0.05 = 0.1.$

$\text{ie : } x_1 = 0.05, x_2 = 0.1 \text{ and } y_{((1),2)} = 6.40625 \times 10^{-5}.$

$$y_{((2),2)} = y_{((1),2)} + \frac{h}{2}\left[f\left(x_1, y_{((1),2)}\right) + f\left(x_2, y_{((2),1)}\right)\right]$$

$$y_{((2),2)} = y_{((1),2)} + \frac{h}{2}\left[\left(x_1^2 + y_{((1),2)}\right) + \left(x_2^2 + y_{((2),1)}\right)\right]$$

$$y_{((2),2)} = 6.40625 \times 10^{-5} + \left(\frac{0.05}{2}\right)[((0.05)^2 + 6.40625 \times 10^{-5}) + ((0.1)^2 + 3.829707031 \times 10^{-4})]$$

$$= 6.40625 \times 10^{-5} + (0.025)[(0.0025 + 6.40625 \times 10^{-5}) + (0.01 + 3.829707031 \times 10^{-4})]$$

$$= 6.40625 \times 10^{-5} + (0.025)(1.29470332 \times 10^{-2})$$

$$= 6.40625 \times 10^{-5} + 3.236758301 \times 10^{-4}$$

$\text{ie } y_{((2),2)} = 3.877383301 \times 10^{-4}$

$$\text{thus; } y_{((2),2)} \cong 0.0004 \tag{16}$$

$\text{at } h = 0.05, i = 2, n = 0, f\left(x_2, y_{((2),2)}\right) = x_2^2 + y_{((2),2)},$

$y_{((2),2)} = 3.877383301 \times 10^{-4}, x_2 = x_1 + h = 0.05 + 0.05 = 0.1,$

$\text{ie } x_2 = 0.1$

$$y_{((3),0)} = y_{((2),2)} + hf\left(x_2, y_{(2,2)}\right)$$

$$y_{((3),0)} = y_{((2),2)} + h\left(x_2^2 + y_{((2),2)}\right)$$

$$= 3.877383301 \times 10^{-4} + (0.05)[(0.1)^2 + 3.877383301 \times 10^{-4}]$$

$$= 3.877383301 \times 10^{-4} + (0.05)[(1.038773833 \times 10^{-2})]$$

$$= 3.877383301 \times 10^{-4} + 5.193869165 \times 10^{-4}$$

$$= 9.071252466 \times 10^{-4}$$

$$\text{thus; } y_{((3),0)} \cong 0.0009 \quad (17)$$

$$\text{at } h = 0.05, i = 2, n = 1, f\left(x_2, y_{((2),2)}\right) = x_2^2 + y_{((2),2)},$$

$$f\left(x_3, y_{((3),0)}\right) = x_3^2 + y_{((3),0)}, x_2^2 + y_{((2),2)} = 1.038773833 \times 10^{-2}$$

$$y_{((2),2)} = 3.877383301 \times 10^{-4}, y_{((3),0)} = 9.071252466 \times 10^{-4}$$

$$x_2 = x_1 + h = 0.05 + 0.05 = 0.1, x_3 = x_2 + h = 0.1 + 0.05 = 0.15$$

$$\text{ie } x_3 = 0.15 \text{ and } x_2 = 0.1.$$

$$y_{((3),1)} = y_{((2),2)} + \frac{h}{2}\left[f\left(x_2, y_{((2),2)}\right) + f\left(x_3, y_{((3),0)}\right)\right]$$

$$y_{((3),1)} = y_{((2),2)} + \frac{h}{2}\left[\left(x_2^2 + y_{((2),2)}\right) + \left(x_3^2 + y_{((3),0)}\right)\right]$$

$$y_{((3),1)} = 3.877383301 \times 10^{-4} + \left(\frac{0.05}{2}\right)[(1.038773833 \times 10^{-2}) + ((0.15)^2 + 9.071252466 \times 10^{-4})]$$

$$= 3.877383301 \times 10^{-4} + (0.025)[(1.038773833 \times 10^{-2}) + (0.0225 + 9.071252466 \times 10^{-4})]$$

$$= 3.877383301 \times 10^{-4} + (0.025)(3.379486358 \times 10^{-2})$$

$$= 3.877383301 \times 10^{-4} + 8.448715894 \times 10^{-4}$$

$$\text{ie } y_{((3),1)} = 1.23260992 \times 10^{-3}$$

$$\text{thus; } y_{((3),1)} \cong 0.0012 \quad (18)$$

$$\text{at } h = 0.05, i = 2, n = 2, f\left(x_2, y_{((2),2)}\right) = x_2^2 + y_{((2),2)},$$

$$f\left(x_3, y_{((3),1)}\right) = x_3^2 + y_{((3),1)}, x_2^2 + y_{((2),2)} = 1.038773833 \times 10^{-2}$$

$$y_{((2),2)} = 3.877383301 \times 10^{-4}, y_{((3),1)} = 1.23260992 \times 10^{-3}$$

$$x_2 = x_1 + h = 0.05 + 0.05 = 0.1, x_3 = x_2 + h = 0.1 + 0.05 = 0.15$$

$$\text{ie } x_3 = 0.15 \text{ and } x_2 = 0.1.$$

$$y_{((3),2)} = y_{((2),2)} + \frac{h}{2}\left[f\left(x_2, y_{((2),2)}\right) + f\left(x_3, y_{((3),1)}\right)\right]$$

$$y_{((3),2)} = y_{((2),2)} + \frac{h}{2}\left[\left(x_2^2 + y_{((2),2)}\right) + \left(x_3^2 + y_{((3),1)}\right)\right]$$

$$y_{((3),2)} = 3.877383301 \times 10^{-4} + \left(\frac{0.05}{2}\right)[(1.038773833 \times 10^{-2}) + ((0.15)^2 + 1.23260992 \times 10^{-3})]$$

$$= 3.877383301 \times 10^{-4} + (0.025)[(1.038773833 \times 10^{-2}) + (0.0225 + 1.23260992 \times 10^{-3})]$$

$$= 3.877383301 \times 10^{-4} + (0.025)(3.412034825 \times 10^{-2})$$

$$= 3.877383301 \times 10^{-4} + 8.530087062 \times 10^{-4}$$

ie $y_{((3),2)} = 1.240747036 \times 10^{-3}$

thus; $y_{((3),2)} \cong 0.0012$ (19)

at $h = 0.05, i = 3, n = 0, f\left(x_3, y_{((3),2)}\right) = x_3^2 + y_{((3),2)},$

$x_3 = 0.15$ and $x_2 = 0.1, y_{((3),2)} = 1.240747036 \times 10^{-3}$

$$y_{((4),0)} = y_{((3),2)} + hf(x_3, y_{(3,2)})$$

$$y_{((4),0)} = y_{((3),2)} + h\left(x_3^2 + y_{((3),2)}\right)$$

$$= 1.240747036 \times 10^{-3} + (0.05)[(0.15)^2 + 1.240747036 \times 10^{-3}]$$

$$= 1.240747036 \times 10^{-3} + (0.05)[(0.0225 + 1.240747036 \times 10^{-3})]$$

$$= 1.240747036 \times 10^{-3} + (0.05)[(2.374074704 \times 10^{-2})]$$

$$= 1.240747036 \times 10^{-3} + 1.187037352 \times 10^{-4}$$

$$= 2.427784388 \times 10^{-3}$$

thus; $y_{((4),0)} \cong 0.0024$ (20)

at $h = 0.05, i = 3, n = 1, f\left(x_3, y_{((3),2)}\right) = x_3^2 + y_{((3),2)},$

$f\left(x_4, y_{((4),0)}\right) = x_4^2 + y_{((4),0)}, x_3^2 + y_{((3),2)} = 2.374074704 \times 10^{-2}$

$y_{((4),0)} = 2.427784388 \times 10^{-3}, y_{((3),2)} = 1.240747036 \times 10^{-3}$

$x_2 = x_1 + h = 0.05 + 0.05 = 0.1, x_3 = x_2 + h = 0.1 + 0.05 = 0.15$

$x_4 = x_3 + h = 0.15 + 0.05 = 0.2$ ie $x_3 = 0.15$ and $x_4 = 0.2.$

$$y_{((4),1)} = y_{((3),2)} + \frac{h}{2}\left[f\left(x_3, y_{((3),2)}\right) + f\left(x_4, y_{((4),0)}\right)\right]$$

ie: $y_{((4),1)} = y_{((3),2)} + \frac{h}{2}\left[\left(x_3^2 + y_{((3),2)}\right) + \left(x_4^2 + y_{((4),0)}\right)\right]$

$$= 1.240747036 \times 10^{-3} + \left(\frac{0.05}{2}\right)[((0.15)^2 + 1.240747036 \times 10^{-3}) + ((0.2)^2 + 2.427784388 \times 10^{-3})]$$

$$= 1.240747036 \times 10^{-3} + (0.025)[(0.0225 + 1.240747036 \times 10^{-3}) + (0.04 + 2.427784388 \times 10^{-3})]$$

$$= 1.240747036 \times 10^{-3} + (0.025)(6.616853142 \times 10^{-2})$$

$$= 1.240747036 \times 10^{-3} + 1.654213286 \times 10^{-3}$$

ie $y_{((4),1)} = 2.90663972 \times 10^{-3}$

thus; $y_{((4),1)} \cong 0.0029$ (21)

at $h = 0.05, i = 3, n = 2, f\left(x_3, y_{((3),2)}\right) = x_3^2 + y_{((3),2)},$

$f\left(x_4, y_{((4),1)}\right) = x_4^2 + y_{((4),1)}, x_3^2 + y_{((3),2)} = 2.374074704 \times 10^{-2}$

$y_{((4),1)} = 2.90663972 \times 10 -^{-3}, y_{((3),2)} = 1.240747036 \times 10^{-3}$

$x_2 = x_1 + h = 0.05 + 0.05 = 0.1, x_3 = x_2 + h = 0.1 + 0.05 = 0.15$

$x_4 = x_3 + h = 0.15 + 0.05 = 0.2$ ie $x_3 = 0.15$ and $x_4 = 0.2.$

$$y_{((4),2)} = y_{((3),2)} + \frac{h}{2}\left[f\left(x_3, y_{((3),2)}\right) + f\left(x_4, y_{((4),1)}\right)\right]$$

$$y_{((4),2)} = y_{((3),2)} + \frac{h}{2}\left[\left(x_3^2 + y_{((3),2)}\right) + \left(x_4^2 + y_{((4),1)}\right)\right]$$

$$y_{((4),2)} = 1.240747036 \times 10^{-3} + \left(\frac{0.05}{2}\right)[((0.15)^2 + 1.240747036 \times 10^{-3}) + ((0.2)^2 + 2.90663972 \times 10^{-3})]$$

$$= 1.240747036 \times 10^{-3} + (0.025)[(0.0225 + 1.240747036 \times 10^{-3}) + (0.04 + 2.90663972 \times 10^{-3})]$$

$$= 1.240747036 \times 10^{-3} + (0.025)(6.664738676 \times 10^{-2})$$

$$= 1.240747036 \times 10^{-3} + 1.66184669 \times 10^{-3}$$

ie $y_{((4),2)} = 2.906931705 \times 10^{-3}$

thus; $y_{((4),2)} \cong 0.0029$ (22)

Table 1: Result Generated from Modified Euler Method (MEM) for the step size h=0.05.

n	x_n	y_n	Analytical Solution	Associated Error (AE)
1	0.05	0.0001	0.0000	0.0001
2	0.1	0.0004	0.0003	0.0001
3	0.15	0.0012	0.0012	0.0000
4	0.2	0.0029	0.0028	0.0001

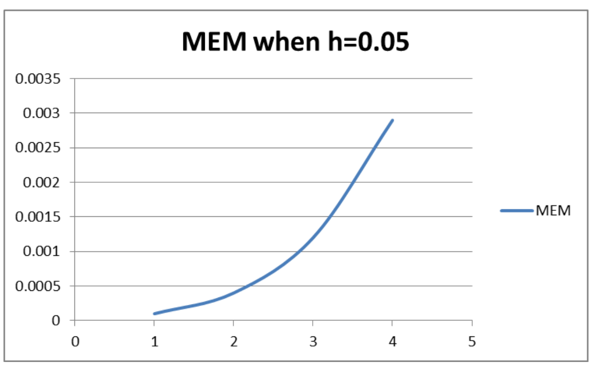

Figure 1: Graphical Solution given by MEM.

ANALYTICAL SOLUTION OF THE PROBLEM

The equation considered in this scope can also be solved through the analytical method using the method of integrating factor as follows:
By the equation described in Problem (1):

Problem 2

Find the values of y(0.1) and y(0.2) from the given differential equation below:

$$\frac{dy}{dx} = x^2 + y$$

with initial condition $y(0) = 0$.

Also find the values of y(0.1) and y(0.2)

Solution 2

Given that $\frac{dy}{dx} = x^2 + y$

Using the method of integrating factor the solution to the given problem as described in Equation (1) is given below:

$$\text{ie: } \frac{dy}{dx} = x^2 + y \equiv y' - y = x^2 \quad (23)$$

$$\Rightarrow \frac{dy}{dx} - y = x^2 \equiv y' - y = x^2, \text{where } y' = \frac{dy}{dx}, p(x) = (-1)$$

$q(x) = x^2$ and integrating factor $(\text{I.F}) = e^{\int p(x)dx}$

$$\text{by the } \mathbf{I.F} = e^{\int p(x)dx}$$
$$= e^{\int (-1)dx}$$
$$= e^{-\int dx}$$
$$= e^{-x}$$

$$\therefore \quad \mathbf{I.F} = e^{-x} \quad (24)$$

now; multiplying equation (23) by equation (**24**)

ie: equation (23) becomes; $y'e^{-x} - ye^{-x} = x^2e^{-x}$

$$\Rightarrow y'e^{-x} - ye^{-x} = \frac{d(ye^{-x})}{dx} = x^2e^{-x}$$

$$\text{ie: } \frac{d(ye^{-x})}{dx}dx = x^2e^{-x}dx$$

$$\text{ie: } d(ye^{-x}) = (x^2e^{-x})dx$$

integratung both sides:

$$\int d(ye^{-x}) = \int (x^2e^{-x})dx$$

$$\Rightarrow ye^{-x} = \int (x^2e^{-x})dx \quad (25)$$

applying method of integration by part to the R.H.S

$$\int udv = uv - \int vdu, \quad (26)$$

where u = function to be differentiated and

$$v = \text{function to be integrated}$$

$$\text{ie: } \int (x^2 e^{-x})dx = RHS \tag{27}$$

$$\text{where } dv = e^{-x}dx \text{ and } v = \int dv = v$$

$$\text{ie: } v = \int e^{-x}dx = [e^{-x}] \div \frac{d(-x)}{dx} = \frac{e^{-x}}{-1} = -e^{-x}$$

$$\text{hence; } v = -e^{-x}$$

$$\text{again by } u = x^2 \Rightarrow \frac{du}{dx} = \frac{d(x^2)}{dx} \Rightarrow du = (x^2)dx$$

$$\Rightarrow du = 2(x^{2-1}) = 2xdx$$

$$\therefore\ v = -e^{-x}, u = x^2, dv = e^{-x} \text{and } du = 2xdx \tag{28}$$

Substituting Equation (28 into (26) to give the point process integral solution of (25)

solution of (25)

$$\text{ie: } \int udv = uv - \int vdu$$

$$\begin{aligned} \Rightarrow \int (x^2 e^{-x})dx &= -x^2 e^{-x} - \int (2x)(-e^{-x})dx \\ &= -x^2 e^{-x} - (-) \int (2x)(e^{-x})dx \\ &= -x^2 e^{-x} + \int (2x)(e^{-x})dx \\ &= -x^2 e^{-x} + \int 2xe^{-x}dx \end{aligned} \tag{29}$$

$$\left.\begin{aligned} &\text{again; } u = 2x, dv = e^{-x}dx \\ &\text{ie: } \frac{du}{dx} = \frac{d(2x)}{dx} \Rightarrow du = (2x)dx \\ &\Rightarrow du = 1.(2x^{1-1}) \\ &= 1.2x^0 \\ &= 1 \times 2 \times 1dx \\ &= 2.dx \\ &= 2dx \end{aligned}\right\} \tag{30}$$

$$\therefore\ v = -e^{-x}, u = x^2, dv = e^{-x} \text{and } du = 2xdx \tag{31}$$

Using Equation 26

$$\text{ie: } \int udv = uv - \int vdu,$$

$\Rightarrow$ equation (43) becomes:

$$= -x^2e^{-x} + \left(uv - \int vdu\right) \quad (32)$$

$$\text{where } v = -e^{-x}, u = 2x, dv = e^{-x} \text{and } du = 2dx \quad (33)$$

$\therefore$ by substituting equation (33) into equation (32)we obtain equation (34):

$$\Rightarrow \int udv = \int 2xe^{-x}dx = -2xe^{-x} - \int (2dx)(-e^{-x})$$

$$\Rightarrow \int (x^2e^{-x})dx = -x^2e^{-x} + \left(-2xe^{-x} - \int (2dx)(-e^{-x})\right)$$

$$\Rightarrow \int (x^2e^{-x})dx = -x^2e^{-x} - xe^{-x} - (-)2\int (e^{-x})\, dx$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\int (e^{-x})\, dx$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\left([e^{-x}] \div \frac{d(-x)}{dx}\right) + C$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\left(\frac{e^{-x}}{-1}\right) + C$$

$$= -x^2e^{-x} - 2xe^{-x} + 2(-e^{-x}) + C$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$= -(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C$$

$$\text{thus; } \int (x^2e^{-x})dx = -(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C$$

$$\Rightarrow \text{by } (3.31): ye^{-x} = \int (x^2e^{-x})dx = -(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C$$

$$\text{ie: } ye^{-x} = -(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C$$

$$\text{ie: } \frac{ye^{-x}}{e^{-x}} = \frac{-(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C}{e^{-x}}, \text{ie dividing both side by } (e^{-x})$$

$$\Rightarrow y = \frac{-(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C}{e^{-x}} = -\left(\frac{x^2e^{-x}}{e^{-x}} + \frac{2xe^{-x}}{e^{-x}} + \frac{2e^{-x}}{e^{-x}}\right) + \frac{C}{e^{-x}}$$

$$\therefore y(x) = Ce^{x} - (x^2 + 2x + 2) \quad (34)$$

Equation (34) gives the equivalence analytical solution of Problem (1, 2, and 3):

But by the given IVP; ie $y(0) = 0, \Rightarrow y = 0$ when $x = 0$

Now substituting the IVP into Equation 34 to obtain the value of the constant term of integration (C)

$\therefore y(0) = -((0)^2 + 2(0) + 2) + Ce^0 = -2 + C \times 1 = 0$

ie: $C - 2 \ = 0, \Rightarrow C + 2 - 2 = 0 + 2, \Rightarrow C = 2$

$$\text{hence } C = 2 \tag{35}$$

$\therefore$ equation (22) becomes :

$y(x) = -(x^2 + 2x + 2) + 2e^x$

$$\text{thus:} y(x) = 2e^x - (x^2 + 2x + 2) \tag{36}$$

REMARK

Equation (36) gives the general non-numerical solution of Problem (2) for any given value of x. Below are the analytical computation of the equivalence unknown solution given by Equation (40):

$ie{:}\ y(x) = 2e^x - (x^2 + 2x + 2)$

so when $x = 0.05$

$$\begin{aligned} ie{:}\ y(0.05) &= 2e^{(0.05)} - ((0.05)^2 + 2(0.05) + 2) \\ &= 2(1.051271096) - (0.0025 + 0.1 + 2) \\ &= 2.102542192 - (2.1025) \\ &= 4.2192 \times 10^{-5} \end{aligned}$$

$$hence \ \ y(0.05) \ \cong 0 \tag{37}$$

$when\ x = 0.1$

$ie\ by{:}\ y(x) = 2e^x - (x^2 + 2x + 2)$

$$\begin{aligned} \Rightarrow \ y(0.05) &= 2e^{(0.1)} - ((0.1)^2 + 2(0.1) + 2) \\ &= 2(1.105170918) - (0.01 + 0.2 + 2) \\ &= 2.210341836 - (2.21) \\ &= 3.41836 \times 10^{-4} \end{aligned}$$

$$hence \ \ y(0.1) \ \cong 0.0003 \tag{38}$$

$when\ x = 0.15$

$ie\ by{:}\ y(x) = 2e^x - (x^2 + 2x + 2)$

$$\begin{aligned} \Rightarrow \ y(0.15) &= 2e^{(0.15)} - ((0.15)^2 + 2(0.15) + 2) \\ &= 2(1.161834243) - (0.0225 + 0.3 + 2) \\ &= 2.323668486 - (2.3225) \\ &= 1.168486 \times 10^{-3} \end{aligned}$$

$$hence \;\; y(0.15) \cong 0.0012 \qquad (39)$$

$$when \; x = 0.2$$

$$ie \; by{:} \; y(x) = 2e^{x} - (x^2 + 2x + 2)$$

$$\Rightarrow \; y(0.15) = 2e^{(0.2)} - ((0.2)^2 + 2(0.2) + 2)$$
$$= 2(1.221402758) - (0.04 + 0.4 + 2)$$
$$= 2.442805516 - (2.44)$$
$$= 2.805516 \times 10^{-3}$$

$$\text{hence} \;\; y(0.2) \cong 0.0028 \qquad (40)$$

Table 2. Result Generated from Analytical (AM) for the step size h =0.05.

n	x_n	Exact Solution(ES)
1	0.05	0.0000
2	0.1	0.0003
3	0.15	0.0012
4	0.2	0.0028

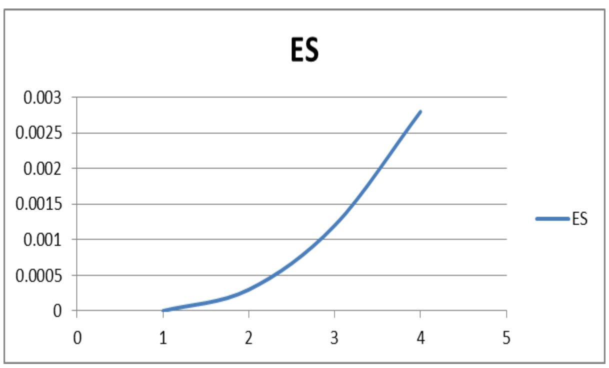

Figure 2: Graphical Solution given by ES

Table 3. Associated Error (AE) for the step size h=0.05.

Associated Error (AE)
0.0001
0.0001
0.0000
0.0001

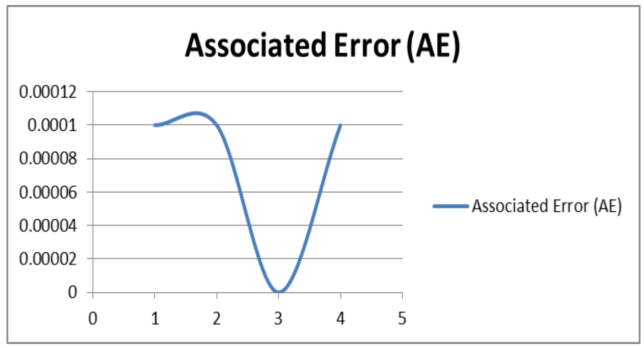

Figure 3: Graphical Solution given by AE.

Table 4: Result Generated from Modified Euler Method (MEM) for the step size h=0.05 and Associated Error (AE).

n	y_n	Associated Error (AE)
1	0.0001	0.0001
2	0.0004	0.0001
3	0.0012	0.0000
4	0.0029	0.0001

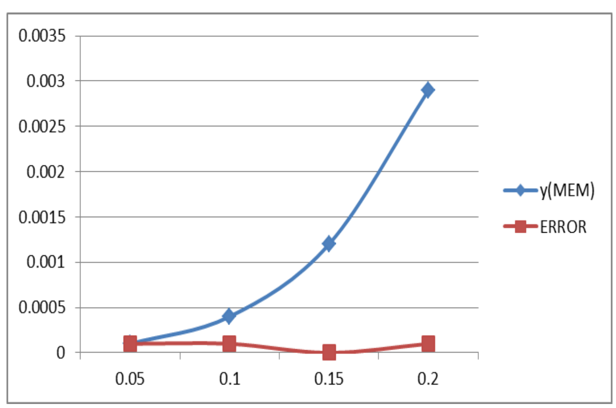

Figure 4: Graphical Solution by Modified Euler Method (MEM) relative to the Associated Error (AE).

SUMMARY OF DERIVED EQUATIONS

From Problem (1), Modified Euler Method (MEM) was used to solve the First Order Differential Equation (FODE) in chapter three as now in (41):

$$\frac{dy}{dx} = x^2 + y \tag{41}$$

with initial condition

$y(0) = 0.$ Also find the values of $y(0.1)$ and $y(0.2)$

$$y_1 = y_1 + \int_{x_0}^{x} f(x,y)\, dx \tag{42}$$

$$y(x) = y(x_0) + \frac{h}{2}\left[f\big(x_0, y(x_0)\big) + f\big(x_1, y(x_1)\big)\right] \tag{43}$$

$$y_1(x_1) = y(x_0) + \frac{h}{2}\left[f\big(x_0, y(x_0)\big) + f\big(x_1, y^0(x_1)\big)\right] \tag{44}$$

$$\Rightarrow y_1^{(0)} = y_0 + hf(x_0, y_0) \tag{45}$$

$$ie\ y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^0)\right] \tag{46}$$

$$ie\ y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^1)\right] \tag{47}$$

$$ie\ y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\big(x_1, y_1^k\big)\right] \tag{48}$$

$$\left.\begin{array}{c} \Rightarrow y_{i+1}^{(0)} = y_i + hf(x_i, y_i) \\ ie\ y_{((i+1),n)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f\left(x_{i+1}, y_{((i+1),n)}\right)\right] \\ f0r\ i = n = 0,1,2,3,\ldots \end{array}\right\} \tag{49}$$

Equations (42) to (53) give the required solution of Equation (41) for four iterations, three sub iterations following for each of one to four iterations (i.e., for n = 0,1 and 2,i = 0, 1, 2, and 3) for the step size h=.05. Thus; the solutions obtained for the problem described in (41) through (49) numerically are:

$$y_{((1),0)} = 0 \tag{42}$$

$$y_{((1),1)} \cong 0.0001(4d.p) \tag{43}$$

$$y_{((1),2)} \cong 0.0001(4d.p) \tag{44}$$

$$y_{((2),0)} \cong 0.0002\ (4d.p) \tag{45}$$

$y_{((2),1)} \cong 0.0004$ (46)

$y_{((2),2)} \cong 0.0004$ (47)

$y_{((3),0)} \cong 0.0009$ (48)

$y_{((3),1)} \cong 0.0012$ (49)

$y_{((3),2)} \cong 0.0012$ (50)

$y_{((4),0)} \cong 0.0024$ (51)

$y_{((4),1)} \cong 0.0029$ (52)

$y_{((4),2)} \cong 0.0029$ (53)

Furthermore, Exact Solution (ES) was also applied in solving Problem (1) and solution was obtained for the two given points of $x\ (ie; x = 0.1\ and\ x = 0.2)$ as required. Equations (54) and (55) give the equivalence numerical solution of Problem (1)

$$y(x) = Ce^{x} - (x^2 + 2x + 2) \quad (54)$$

$$y(x) = 2e^{x} - (x^2 + 2x + 2) \quad (55)$$

More so, the resulting numerical solution was obtained. See Equations (56) to (59) for the ranges of values of x: ($0.05 \leq x_n \leq 0.2$) ∀ n= 1, 2, 3….below are the numerical result obtained for the Exact Solution. (ES).

$y(0.05) \cong 0$ (56)

$y(0.1) \cong 0.0003$ (57)

$y(0.15) \cong 0.0012$ (58)

$y(0.2) \cong 0.0028$ (59)

RESULTS AND DISCUSSION

In Equations (9)-(10) and (36) shows the derived general form of the Modified Euler Analytical Method (AM), respectively. Also, Equations (11) – (22) gives the approximate numerical solution to four decimal place of Problem (1 and 2) using the proved Equations (9)-(10) and (36) Modified Euler and AM. Again, Equations (23) – (36) give the solution of Problem 2 by analytical approach using the concept of integrating factor and numerical solution for the analytical method of the general solution of Problem 2 were computed for the defined range of values of x as equated in Equations (37)-(40) through the use of Equation (36). In addition, graphical illustrations for the general solutions and associated error were shown and displayed in Figures 1 to 4 for the approach and techniques applied (Modified Euler and Analytical Method (AM)).

Tables 1 – 4 show the numerical results together with their associated errors where necessary for solutions obtained from solutions of the Problems

(1 and 2) using Modified Euler Method as well as the Analytical Method. Table 1 shows the numerical solution obtained from Modified Euler Method (MEM) for the successive iterations. More so, numerical solution from the Analytical Method and the associated error were also displayed and the associated error were also displayed. Table 2 shows the numerical solution generated from Analytical Method (AM) as the Exact Solution for the successive iterations.

MEM gives 20% [14]. Hence it is still evident to say that MEM is more accurate than EM by average percentage error [14]. Furthermore, Figure 1 shows the nature of the numerical solution iterated from Modified Euler. In Figure 2. Similarly, shows the graphical nature and graphical solution of Problem 2 relative to the solution from Analytical Method (AM). In Figure 2 graphical nature of associated error was graphically demonstrated in the numerical solution of Problem 1 and 2.

More so, the graphical illustration in Figure 4 displays the distinction and uniqueness by associated errors in the numerical solutions of the Problem 1 using MEM. It is clear to say that; MEM most accurate than EM graphically relating the path nature of the two graphs of MEM and EM [14] for the solution of FODE's. This is because the numerical Scheme (MEM) gives closely or approximately the same solution of the problem as the solution generated from the Analytical Method (i.e., the ES).

The graphical solutions helps to show how unique by either convergence or associated error a given method is, as such comparing the line curve for the solution of the problem in Figure 1; the graphical solutions displayed for MEM as well as the results in Table 1 to 2 , it was observed that MEM can further be investigated in terms of Accuracy, iteration process, rate of convergence, associated error with other methods excluding EM and PM but numerical schemes such Adams-Bashfort and so on.

CONCLUSION

Explicitly and analytically solution has been obtained for the problem considered for both methods and justified numerically; as such it is very important to conclude that MEM is a Stable Numerical Scheme (SNS) for the solution of First Order Differential Equation (FODE) with Initial Value Problem (IVP) with negligibly zero Globally Associated Error (GAE) and converges slowly but guaranteed for convergence. This is because Modified Euler Method (MEM) was investigated and discovered more accurate than Euler Method (EM) [14] with less Globally Associated Error (GAE) but requires multiple iterations before converging to the approximate solution at the given point of evaluation than Euler Method (EM). Thus by the aim and objective of this paper, successful conclusion is said that MEM is a CSNS, more accurate, most Convergent, Stable and reliable for the solution of First order Differential Equations (FODE's) with Initial Value Problem (IVP) over Euler Method (MEM) with respect to the Analytical Method (AM).

REFERENCES

1. Parker, G.E. and J.S. Sochacki. 1996. "Implementing the Picard Iteration". *Neural, Parallel, and Scientific Computation* 3(2):1271-1275, 2013 4.

2. Parker, G.E. and J.S. Sochacki. 2000. "A Picard-McLaurin Theorem for Initial Value PDE's". *Abstract and Applied Analysis*.5(1):47–63.

3. Crowel, B.J. 1986. *Higher Engineering Mathematical*. Prentice Hall: New Deli, India.

4. Edalat, A.P. 2007. "D.A Domain Theoretic Account of Picard". *LMS Journal of Computation and Mathematics*. 10:83-118.

5. Garrett, B. and R. Gian- Carlo. 1978. *Ordinary Differential Equations, 3rd Ed*. John Wiley and Sons: New York, NY.

6. Einar, H. 1969. *Lectures on Ordinary Differential Equations*. Addison-Wesley Pub. Co.: Reading, MA.

7. Paniconi, C. and M. Putti. 1994. "A Comparison of Picard and Newton Iteration in the Numerical Solution of Multidimensional Variably Saturated flow Problems". *Water Resour. Res.* 30(12): pp3357-337 doi: 1029/94WR02046.

8. Samuel. G. 1954. *Introduction to Differential Equation*. John Wiler and Sons Inc.: New York, NY.

9. Haye, S.I. 2001. *Advance Mathematics Method for Science and Engineering Mathematics*. Marcel Dekker, Inc.: New York, NY.

10. Kawamura, A. 2009. *Lipchitz Continuous Ordinary Differential Equations and Polynomial Space Complete*. In: CCC '09: 24th Annual IEEE Conference on Computational Complexity. 1:149-160.

11. Mark, A.P. 1984. *Introduction to Partial Differential Equation with Application*. McGraw Hill: New York, NY.

12. Shawagfeh, N. and D. Kaya. 2004. "Comparing Numerical Methods for the Solutions of Systems of Ordinary Differential Equations". *Appl Math Lett*. 17:323-328.

13. Carothers, D.C., G.E. Parker, J.S. Sochacki, and P.G. Warne. 2005. "Some Properties of Solutions of Polynomial Systems of Differential Equations". *Electronic Journal of Differential Equations*. 40:1-17.

14. Lanlege, D.I., A.A. Wachin, U.M Garba, and A. Aluebho. 2015. "Comparison of Numerical Solution of Some First Order Differential Equations (FODE's) with Initial Value Problems (IVP's) Using Picard, Euler and Modified Euler Methods (PEMEM)". 29(2):29-48.

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