On Linear Transformation in Linear Orthogonality Spaces.

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ABSTRACT

Let (E, $\perp$) be linearly orthogonality space, f is orthogonally additive and $X, Y \in E$. It is proved that, under suitable assumptions on E, f is sublinear functional. Furthermore, if $T: X \longrightarrow Y$ is orthogonally additive, then T^{-1} is continuous.

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INTRODUCTION

Many researchers have studied extensively the concept of orthogonality additive mappings from an orthogonality spaces. In 1995, Baron and Ratz [1] proved that if (Y,+) is an abelian group and $f: X \rightarrow Y$ is orthogonally additive, then there exists exactly one pair of additive mapping $a: \mathbb{R} \rightarrow Y, b: X \rightarrow Y$ such that:

$$f(x) = a\left(||x||^2\right) + b(x), x \in X.$$

In [6], James showed that isosceles and Pythagorean orthogonality are equivalent in real Hilbert spaces and are also symmetric, additive and homogeneous. However, they are not additive and homogeneous in general normed linear spaces. In 2002, Saidi discussed the characterization of orthogonality in certain Banach spaces and an extension of the notion of orthogonality to Banach spaces.

Wlodzimier and Justyna (2008) discussed sandwich theorems for orthogonally additive functions. If p and q are orthogonally subadditive mappings such that $p \leq q$ or $q \leq p$, they show that under some additional assumptions, there exists a unique orthogonally additive mapping f such that $p \leq f \leq q$ or $q \leq f \leq p$, respectively.

In [8], Reiz Representation Theorem was given where f is bounded linear fractional. The theorem can be extended to linear orthogonality spaces via orthogonally additive and odd functional; let $(E, \perp)$ be a linear orthogonality normed space and E a Hilbert space. Let $f: E \rightarrow \mathbb{R}$ be orthogonally and odd. Then, there exists a unique vector $y_0 \in E$ such that $f(x) = \langle x, y_0 \rangle$ for each $x \in E$ and moreover, $||f|| = ||y_0||$. See Kuczma (2009) for the theory of functional equations and inequalities.

For relevant work on orthogonally additive, orthogonal increasing functions, orthogonally exponential functional, orthogonality and linear functional in normed linear spaces, see Birkhoff (1935), Gudder and Strawther (1975), Dragomir and Koliha (1991), and Bkrzde (1997).

In this work, some results on operators of normed linear spaces are extended to linear orthogonality spaces. Throughout the article, $\mathbb{R}$ denotes the set of all real numbers, $\{\mathbb{R}_+ = t \in \mathbb{R}: t \geq 0\}$ = and $\mathbb{C}$ the set of all complex numbers.

PRELIMINARY RESULTS

1. Definition (Linearity Orthogonality Space)

A pair $(E, \perp)$ is linear orthogonality space provided that E is a real linear space with dim $E \geq 2$ and $\perp \subset E^2$ is a relation such that:

(L01) $x \perp y$ if and only if $y \perp x$ and $x \perp y$ for every $x \in E$ implies $y = 0$ or $y \perp x$ for all $y \in E$ implies $x = 0$

(L02) if $x, y \in E \setminus \{0\}$ and $x \perp y$, then x and y are linearly independent,

(L03) if $x, y, z \in E$, $x \perp y$ and $x \perp z$ imply $x \perp y + z$

(L04) if $x, y \in E$ and $x \perp y$, then $ax \perp by$ for every $a, b \in \mathbb{R}$,

(L05) if P is a two dimensional subspace of E, $x \in P$ and $\alpha \in \mathbb{R}^+$, then there is $y \in P$ with $x \perp y$ and $x + y \perp \alpha x - y$.

Any linear space can be made into a linear orthogonality space if we define by $x \perp 0$, $0 \perp x$ for all x, and for non-zero vectors x, y defined by $x \perp y$ if and only if x, y are linearly independent.

2. Definition (Orthogonally Additive Functions)

A real-valued function $f: E \to \mathbb{R}$ on a normed linear space E is orthogonally additive if and only if $f(x+y) = f(x) + f(y)$ whenever $x \perp y \ \forall\ x, y \in E$.

A linearly orthogonality space endowed with a norm is called a linear orthogonality normed space. In [7] the following result among others was discussed:

3. Lemma [7]: Let $(E, \perp)$ be a linear orthogonality normed space and let $f: E \to \mathbb{R}$ such that $f(x) = \| x \|^2$ be orthogonally additive.

(a) If f is odd, then f is linear

(b) If f is even, then $f(\alpha x) = \alpha^2 f(x)$ for all $\alpha \in \mathbb{R}$ and if $x, y \in E$, then $f(x) = f(y)$.

4. Lemma: If $f: E \to \mathbb{R}^+$ is orthogonally additive, then there is a $c \in \mathbb{R}^+$ with $f(x) = c||x||^2$.

Proof: Since $f(x)$ is orthogonally additive function then $cf(x)$ is also orthogonally additive function and the result follows from the properties of additivity on f.

MAIN RESULTS

The following theorem will be useful in the proof of our main results:

5. Theorem

Let $(E, \perp)$ be a linear orthogonality normed space and E a Hilbert space. Let $f: E \to \mathbb{R}$ be orthogonally additive and odd. Then,

(i) There exist a unique vector $y_0 \in E$ such that $f(x) = \langle x, y_0 \rangle$ for each $x \in E$

(ii) Moreover, $||f|| = ||y_0||$

Proof: The proof can be sourced from [7].

6. Theorem

Let ϵ be a linearly orthogonality space. If $f: \epsilon \to \mathbb{R}$ is orthogonally additive and $f(x) \leq M \| x \|^2$ for all $x \in E$ for some $M \geq 0$, then there is an $\alpha \in \mathbb{R}$ such that $f(x) = \alpha \| x \|$.

Proof: Let $h(t) = M \| x \|^2 - f(x)$

Then, $h: \epsilon \to \mathbb{R}^+$ is orthogonally additive function, we have,

$$h(x+y) = M \| x+y \|^2 - f(x+y)$$
$$= M(\| x \|^2 + \| y \|^2) - (f(x) + f(y))$$
(by Definition 3).

$$= M \| x \|^2 + M \| y \|^2 - f(x) - f(y)$$

$$= (M \| x \|^2 - f(x)) + (M \| y \|^2 - f(y))$$
$$= h(x) + h(y)$$

By Lemma 4, there is a $c \geq 0$ such that $h(x) = c \| x \|^2$ Hence from (1),

$c \parallel x \parallel^2 = M \parallel x \parallel^2 - f(x)$
$\Longrightarrow f(x) = M \parallel x \parallel^2 - c \parallel x \parallel^2$
$= (M - c) \parallel x \parallel^2$

Set $\alpha = M - c$. Then, $f(x) = \alpha \parallel x \parallel^2$

7. Theorem

Let E be a linear orthogonality space and let $f: E \to \mathbb{R}$ be orthogonally additive and satisfy $|f(x)| \leq M \parallel x \parallel$ for all $x \in E$. Then, f is sublinear functional.

Proof: If $x \perp y$, then by L04,
$ax \perp by \ \ \forall\, a, b \in \mathbb{R}$

Since, f is orthogonally additive,

$f(ax + by) = f(ax) + f(by)$
$\leq M \parallel ax \parallel + M \parallel by \parallel = a(M \parallel x \parallel) + b(M \parallel y \parallel)$
$f(ax + by) \leq af(x) + bf(y)$

Therefore, f is sublinear.

We need to show that $|f(x)| \leq M \parallel x \parallel$ implies with bound M.

Let $\epsilon > 0$, and let $\delta = \frac{\epsilon}{M}$. If $\parallel x \parallel < \delta$
$\parallel f(x) \parallel \leq M \parallel x \parallel < M\delta$
$\parallel f(x) \parallel \leq M \parallel x \parallel < \epsilon$

So f is continuous at $x = 0$

It follows that if $\parallel x - y \parallel < \delta$, then
$\parallel f(x) - f(y) \parallel = \parallel f(x - y) \parallel < \epsilon$

Which implies that f is continuous at every point on E.

8. Theorem

Let $(X, \perp)$ and $(Y, \perp)$ be linearly orthogonal normed spaces. Suppose that $T: X \to Y$ is orthogonally additive and $T \leq M \parallel x \parallel$ for all $x \in X$, $M \geq 0$.

Then, $T^{-1}: Y \to X$ is continuous.

Proof: Let $x, y \in X$. Since $T$is orthogonally additive, $x \perp y$ implies:

$T(x + y) = T(x) + T(y)$

Let $H(x) = M \parallel x \parallel^2 - T(x)$.

Then by Theorem 6, H is orthogonally additive and $T(x) = f(x) = \alpha \parallel x \parallel^2$.

$T(ax + by) = \alpha(\parallel ax + by \parallel^2)$
$a, b \in \mathbb{R}^+, x, y \in X$

$= \alpha(\parallel ax \parallel^2 + \parallel by \parallel^2)$

by (L04)
$= \alpha \parallel ax \parallel^2 + \alpha \parallel by \parallel$
$= T(ay) + T(by)$

Which implies that $T: X \to Y$ is linear if:

$T(x) = \alpha \parallel x \parallel^2 = 0$. Then, $\parallel x \parallel^2 = 0 \Longrightarrow x = 0$.

Therefore, T is one-to-one, then $\exists\ T^{-1}$ on $R(T)$ and $T^{-1}: R(T) \to X$ is linear.

Let $D := \{y \in Y: \parallel y \parallel \leq 1\}$ be the closed unit disc in Y.

Let y_1 and y_2 be arbitrary element of Y. Assume that $y_1 - y_2 \neq 0$. Consider

$z = \frac{y_1 - y_2}{\parallel y_1 - y_2 \parallel}$, $z \in$ D. Since $z \in$ D, there exists a constant $M \geq 0$ such that,

$\parallel T^{-1}(z) \parallel \leq M$. That is $\left\| T^{-1}\left(\frac{y_1 - y_2}{\parallel y_1 - y_2 \parallel}\right) \right\| \leq M$ or

$\parallel T^{-1} y_1 - T^{-1} y_2 \parallel \leq M \parallel y_1 - y_2 \parallel$

This shows that T^{-1} is bounded and continuous.

9. Corollary

Let X and Y be linearly orthogonality normed spaces. Suppose that $T: X \rightarrow$ Y satisfying $T(x) \leq M \| x \|$ for all $x \in E$ and $M \geq 0$ is:

(i) orthogonality additive;

(ii) bijective;

(iii) continuous;

(iv) linear.

Then, $T^{-1}: Y \rightarrow X$ is continuous.

Proof: Let T be defined as in Corollary 9 and by using the argument in Theorem 8, T is linear and one-to-one.

If T is bijective, then there exists T^{-1}: $Y \rightarrow X$ which is linear. Since T is continuous, then for each $x \in$ X there exists $M \geq 0$ such that $\| Tx \| < M$ with $\| x \| < 1$. This in turn, implies that $\| x \| \leq \frac{1}{M} \| Tx \|$ for each $x \in X$. That is $\| T^{-1}x \| \leq \frac{1}{M} \| x \|$ which implies that T^{-1} is continuous.

CONCLUSION

This article has extended some valid results on operators of normed linear spaces to linear orthogonality spaces.

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