A Zero-Stable Block Method for the Solution of Third Order Ordinary Differential Equations.

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ABSTRACT

In this work, we derived new schemes for $k = 4$, for the solution of third-order Ordinary Differential Equations of the form:

$$y''' = f(x, y, y', y''),\ y(\alpha) = y_0,\ y'(\alpha) = \beta,\ y''(\alpha) = \eta$$

with the associated boundary conditions. The implementation strategies of the derived methods have shown that the block methods are found to be consistent, zero-stable and hence convergent. The derived method was tested on third-order ordinary differential equations, and the numerical results obtained compared favorably with the exact solutions.

(Keywords: block method, linear multistep method, self-starting, special third order, differential equations)

INTRODUCTION

Many methods have been developed for the solution of Ordinary Differential Equations. The speed and the accuracy of a method are the two major characteristics that give any method an upper hand over any other method [23], [24]. Many have undergone changes to either to shed more light on their behavior and their accuracies. In seeking the highest possible order that can be achieved by a linear k-step method, the consistency condition is automatically achieved but by “first Dahlquist barrier”, in attempting to satisfy the root condition. The highest order that can be achieved by a linear k-step method is $2k$ if the method is implicit and $2k - 1$ if the method is explicit. Linear k-step method achieving such orders are called Maximal. However in general, maximal methods which fail to satisfy the root condition are thus Zero-Unstable [11], [12], [23].

In most applications, higher orders are solved by reducing to an equivalent system of first order ordinary differential equation, for which an appropriate numerical method would be employed to solve the resultant system [1] – [10], [16], [17], [19], [21-22], and [25-29].

Ordinary differential equations are often used to describe physical systems. The solution of such equations gives valuable insight into how the system evolves and what the effects of changes in the system are. In general, it is extremely difficult, if not impossible, to obtain an analytic solution. In science and engineering, usually mathematical models are developed to help in the understanding of physical phenomena. These models often yield equations that contain some derivatives of an unknown function of one or several variables. Such equations are called differential equations. Differential equations do not only arise in the physical sciences but also in diverse fields as economics, medicine, psychology, operational research, and even in areas such as biology and anthropology.

Interestingly, differential equations arising from the modeling of physical phenomena often do not have analytic solutions. Hence, the development of numerical methods to obtain approximate solutions becomes necessary. To that extent, several numerical methods such as Finite Difference Methods, Finite Element Methods and Finite Volume Methods, among others, have been developed based on the nature and type of the differential equation to be solved.

A differential equation can simply be defined as an equation that contains a derivative. In other words, it's a relationship involving an independent variable x, a dependent variable y and one or more derivative of y with respect to *x*. An example of a differential equation is:

$$y''' - y'' + y' - y = 0 \qquad (1.1)$$

Differential equations are of two types: An Ordinary Differential Equation (ODE) is one for which the unknown function (also known as dependent variable) is a function of a single independent variable. While, a Partial Differential Equation (PDE) is a differential equation in which the unknown function is a function of multiple independent variables and the equation involves its partial derivatives. An ODE is classified according to the order of the highest derivative with respect to the dependent variable appearing in the equation. The most important cases for applications are the first and second order. In particular, Finite Difference Methods have excelled for the numerical treatment of ordinary differential equations especially since the advent of digital computers. The development of algorithms has been largely guided by convergence theorems Dahlquist [11], [12] as well as the treatises of Henrici and Stetter, Fatunla [13] – [15], [18], and [20].

The development of numerical methods for the solution of Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs) of the form:

$$y^{(\mu)} = f\left(x, y, y^{(1)}, \ldots, y^{(\mu-1)}\right), y(a) = \eta_0, y^{(1)}(a) = \eta_1, \ldots, y^{(\mu-1)}(a) = \eta_{\mu-1} \qquad (1.2)$$

on the interval $[a, b]$ has given rise to two major discrete variable methods namely; one-step (or single step) methods and multistep methods especially the Linear Multistep Methods (LMMs).

A numerical method is a difference equation involving a number of consecutive approximations $y_{n+j}, j = 0,1,2, \ldots, k$ from which it will be possible to compute sequentially the sequence $\{y_n | \, n = 0,1,2, \ldots, N\}$. Naturally, this difference equation will also involve the function *f*. The integer *k* is called the step number of the method.

For $k = 1$, it's called a 1-step method and for value of $k > 1$ it's called a multistep or *k*-step method [23].

One step methods include the Euler's methods, the Runge-Kutta methods, the Theta methods, etc. These methods are only suitable for the solutions of first order IVPs of ODEs because of their very low order of accuracy. In order to develop higher order one step methods such as Runge-Kutta methods, the efficiency of Euler's methods, in terms of the number of functional evaluations per step is sacrificed since more valuations is required. Hence, solving (1.1) using any of the one step methods means reducing it to an equivalent system of first order IVPs of ODEs which increases the dimension of the problem thus increasing its scale. The result is that one step methods become time-consuming for large scale problems and give results that are of low accuracy.

If a computational method for determining the sequence $\{y_n\}$ takes the form of a linear relationship between $y_{n+j}, f_{n+j}, j = 0,1,2, \ldots, k$ we call it a Linear Multistep Method of step number *k* or a Linear *k*-step method. These methods can be written in the general form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^{\mu} \sum_{j=0}^{k} \beta_j f_{n+j} \qquad (1.3)$$

where α_j, β_j are constants and we assume $\mu = (1,2,3, \ldots)$ and $\alpha_k \neq 0$. Without loss of generality, $\alpha_k = 1$ always. Explicit methods are characterized by $\beta_k = 0$ and implicit methods by$\beta_k = 1$.

Explicit linear multistep methods are known as Adams-Bashforth methods, while implicit linear multistep methods are called Adams-Moulton methods. These methods are generally called the Adams family. Other famous classes of multistep methods aside the Adams family includes the Predictor-Corrector method and the Backward Differentiation Formula.

The idea of Taylor's expansion was employed to derive Block Backward Differentiation Formulae (BBDF), for $k = 3$. The derived Hybrid Backward Differentiation Formulae were then used as a Block Backward Differentiation Formulae (BBDF) to solve Initial Value Problems of third-order Ordinary Differential Equations directly without the need of a starting value, which forms the basis of this paper.

MATERIALS AND METHODS

Derivation of the Method

Constructing an implicit linear three-step method of order three of the form (1.3) with $\mu = 3$,

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j}$$

where $k = 4$ and $\alpha_k = \alpha_4 = 1$.

Let $\alpha_0 = a$, be the free parameter.

We now have the general form of the proposed method expressed as:

$$\alpha_4 y_{n+4} + \alpha_3 y_{n+3} + \alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h^3 [\beta_4 y_{n+4} + \beta_3 f_{n+3} + \beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n] \qquad (2.1)$$

Since $\alpha_4 = 1$ and $\alpha_0 = a$ are already known.

Equation (2.1) implies:

$$y_{n+4} + \alpha_3 y_{n+3} + \alpha_2 y_{n+2} + \alpha_1 y_{n+1} + a y_n = h^3 [\beta_4 y_{n+4} + \beta_3 f_{n+3} + \beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n] \qquad (2.2)$$

Then, the remaining undetermined parameters are $\alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3$ and β_4.

Now on using Taylor expansion for (2.2), we have:

$$\begin{aligned} y_{n+4} &= y_n + 4hy_n' + \frac{(4h)^2}{2!}y_n'' + \frac{(4h)^3}{3!}y_n''' + \frac{(4h)^4}{4!}y_n^{iv} + \frac{(4h)^5}{5!}y_n^{v} + \frac{(4h)^6}{6!}y_n^{vi} + \frac{(4h)^7}{7!}y_n^{vii} + \frac{(4h)^8}{8!}y_n^{viii} + \cdots \\ &= y_n + 4hy_n' + 8h^2y_n'' + \frac{32}{3}h^3y_n''' + \frac{32}{3}h^4y_n^{iv} + \frac{128}{15}h^5y_n^{v} + \frac{256}{45}h^6y_n^{vi} + \frac{1024}{315}h^7y_n^{vii} + \frac{512}{315}y_n^{viii} + \cdots \end{aligned} \qquad (2.3)$$

$$\begin{aligned} \alpha_3 y_{n+3} &= \alpha_3\left[y_n + 3hy_n' + \frac{(3h)^2}{2!}y_n'' + \frac{(3h)^3}{3!}y_n''' + \frac{(3h)^4}{4!}y_n^{iv} + \frac{(3h)^5}{5!}y_n^{v} + \frac{(3h)^6}{6!}y_n^{vi} + \frac{(3h)^7}{7!}y_n^{vii} + \frac{(3h)^8}{8!}y_n^{viii} + \cdots\right] \\ &= \alpha_3\left[y_n + 3hy_n' + \frac{9}{2}h^2y_n'' + \frac{9}{2}h^3y_n''' + \frac{27}{8}h^4y_n^{iv} + \frac{81}{40}h^5y_n^{v} + \frac{81}{80}h^6y_n^{vi} + \frac{243}{560}h^7y_n^{vii} + \frac{729}{4480}h^8y_n^{viii} + \cdots\right] \end{aligned} \qquad (2.4)$$

$$\alpha_2 y_{n+2} = \alpha_2\left[y_n + 2hy_n' + \frac{(2h)^2}{2!}y_n'' + \frac{(2h)^3}{3!}y_n''' + \frac{(2h)^4}{4!}y_n^{iv} + \frac{(2h)^5}{5!}y_n^{v} + \frac{(2h)^6}{6!}y_n^{vi} + \frac{(2h)^7}{7!}y_n^{vii} + \frac{(2h)^8}{8!}y_n^{viii} + \cdots\right]$$

$$= \alpha_2 \left[y_n + 2hy_n' + 2h^2 y_n'' + \frac{4}{3}h^3 y_n''' + \frac{2}{3}h^4 y_n^{iv} + \frac{4}{15}h^5 y_n^{v} + \frac{4}{45}h^6 y_n^{vi} + \frac{8}{315}h^7 y_n^{vii} + \frac{2}{315}h^8 y_n^{viii} + \cdots \right] \quad (2.5)$$

$$\alpha_1 y_{n+1} = \alpha_1 \left[y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \frac{h^4}{4!}y_n^{iv} + \frac{h^5}{5!}y_n^{v} + \frac{h^6}{6!}y_n^{vi} + \frac{h^7}{7!}y_n^{vii} + \frac{h^8}{8!}y_n^{viii} + \cdots \right]$$

$$= \alpha_1 \left[y_n + hy_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \frac{h^4}{24}y_n^{iv} + \frac{h^5}{120}y_n^{v} + \frac{h^6}{720}y_n^{vi} + \frac{h^7}{5040}y_n^{vii} + \frac{h^8}{40320}y_n^{viii} + \cdots \right] \quad (2.6)$$

$$ay_n = ay_n \quad (2.7)$$

$$h^3\beta_4 f_{n+4} = h^3\beta_4 \left[y_n' + 4hy_n'' + 8h^2 y_n''' + \frac{32}{3}h^3 y_n^{iv} + \frac{32}{3}h^4 y_n^{v} + \frac{128}{15}h^5 y_n^{vi} + \frac{256}{45}h^6 y_n^{vii} + \frac{1024}{315}h^7 y_n^{viii} + \cdots \right] \quad (2.8)$$

$$h^3\beta_3 f_{n+3} = h^3\beta_3 \left[y_n' + 3hy_n'' + \frac{9}{2}h^2 y_n''' + \frac{9}{2}h^3 y_n^{iv} + \frac{27}{8}h^4 y_n^{v} + \frac{81}{40}h^5 y_n^{vi} + \frac{81}{80}h^6 y_n^{vii} + \frac{243}{560}h^7 y_n^{viii} + \cdots \right] \quad (2.9)$$

$$h^3\beta_2 f_{n+2} = h^3\beta_2 \left[y_n' + 2hy_n'' + 2h^2 y_n''' + \frac{4}{3}h^3 y_n^{iv} + \frac{2}{3}h^4 y_n^{v} + \frac{4}{15}h^5 y_n^{vi} + \frac{4}{45}h^6 y_n^{vii} + \frac{8}{315}h^7 y_n^{viii} + \cdots \right] \quad (2.10)$$

$$h^3\beta_1 f_{n+1} = h^3\beta_1 \left[y_n' + hy_n'' + \frac{h^2}{2} y_n''' + \frac{h^3}{6} y_n^{iv} + \frac{h^4}{24}y_n^{v} + \frac{h^5}{120}y_n^{vi} + \frac{h^6}{720}y_n^{vii} + \frac{h^7}{5040}y_n^{viii} + \cdots \right] \quad (2.11)$$

$$h^3\beta_0 f_n = h^3\beta_0 y_n' \quad (2.12)$$

Comparing equations (2.3) to (2.12), we obtain the followings;

$$1 + \alpha_3 + \alpha_2 + \alpha_1 + a = 0 = C_0$$
$$\alpha_1 + \alpha_2 + \alpha_3 = -1 - a \quad (2.13)$$

$$4 + 3\alpha_3 + 2\alpha_2 + \alpha_1 - \beta_4 - \beta_3 - \beta_2 - \beta_1 - \beta_0 = 0 = C_1$$
$$\alpha_1 + 2\alpha_2 + 3\alpha_3 - \beta_0 - \beta_1 - \beta_2 - \beta_3 - \beta_4 = -4 \quad (2.14)$$

$$8 + \frac{9}{2}\alpha_3 + 2\alpha_2 + \frac{1}{2}\alpha_1 - 4\beta_4 - 3\beta_3 - 2\beta_2 - \beta_1 = 0 = C_2$$
$$\alpha_1 + 4\alpha_2 + 9\alpha_3 - 2\beta_1 - 4\beta_2 - 6\beta_3 - 8\beta_4 = -16 \quad (2.15)$$

$$\frac{32}{3} + \frac{9}{2}\alpha_3 + \frac{4}{3}\alpha_2 + \frac{1}{6}\alpha_1 - 8\beta_4 - \frac{9}{2}\beta_3 - 2\beta_2 - \frac{1}{2}\beta_1 = 0 = C_3$$
$$\alpha_1 + 8\alpha_2 + 27\alpha_3 - 3\beta_1 - 12\beta_2 - 27\beta_3 - 48\beta_4 = -64 \quad (2.16)$$

$$\frac{32}{3}+\frac{27}{8}\alpha_3+\frac{2}{3}\alpha_2+\frac{1}{24}\alpha_1-\frac{32}{3}\beta_4-\frac{9}{2}\beta_3 -\frac{4}{3}\beta_2-\frac{1}{6}\beta_1=0=C_4$$

$$\alpha_1+16\alpha_2+81\alpha_3-4\beta_1-32\beta_2-108\beta_3 -256\beta_4 =-256 \quad (2.17)$$

$$\frac{128}{15}+\frac{81}{40}\alpha_3+\frac{4}{15}\alpha_2+\frac{1}{120}\alpha_1-\frac{32}{3}\beta_4 -\frac{27}{8}\beta_3-\frac{2}{3}\beta_2-\frac{1}{24}\beta_1=0 =C_5$$

$$\alpha_1+32\alpha_2+243\alpha_3-5\beta_1-80\beta_2-405\beta_3 -1280\beta_4 =-1024 \quad (2.18)$$

$$\frac{256}{45}+\frac{81}{80}\alpha_3+\frac{4}{45}\alpha_2+\frac{1}{720}\alpha_1-\frac{128}{15}\beta_4 -\frac{81}{40}\beta_3-\frac{4}{15}\beta_2-\frac{1}{120}\beta_1 =0=C_6$$

$$\alpha_1+64\alpha_2+729\alpha_3-6\beta_1-192\beta_2 -1458\beta_3-6144\beta_4 =-4096 \quad (2.19)$$

$$\frac{1024}{315}+\frac{243}{560}\alpha_3+\frac{8}{315}\alpha_2+\frac{1}{5040}\alpha_1 -\frac{256}{45}\beta_4-\frac{81}{80}\beta_3-\frac{4}{45}\beta_2 -\frac{1}{720}\beta_1=0=C_7$$

$$\alpha_1+128\alpha_2+2187\alpha_3-7\beta_1-448\beta_2 -5103\beta_3-28672\beta_4 =-16384 \quad (2.20)$$

$$\frac{512}{315}+\frac{729}{4480}\alpha_3+\frac{2}{315}\alpha_2+\frac{1}{40320}\alpha_1 -\frac{1024}{315}\beta_4-\frac{243}{560}\beta_3-\frac{8}{315}\beta_2 -\frac{1}{5040}\beta_1=0=C_8$$

$$\frac{1}{40320}(\alpha_1+256\alpha_2+6561\alpha_3-8\beta_1 -1024\beta_2-17496\beta_3-131072\beta_4 +65536) =C_8 \quad (2.21)$$

From (2.13) to (2.21) we obtain a six system of equation given as follows:

$$\alpha_1+\alpha_2+\alpha_3=-1-a$$
$$\alpha_1+2\alpha_2+3\alpha_3-\beta_0-\beta_1-\beta_2-\beta_3-\beta_4 =-4$$
$$\alpha_1+4\alpha_2+9\alpha_3-2\beta_1-4\beta_2-6\beta_3-8\beta_4 =-16$$
$$\alpha_1+8\alpha_2+27\alpha_3-3\beta_1-12\beta_2-27\beta_3 -48\beta_4=-64$$
$$\alpha_1+16\alpha_2+81\alpha_3-4\beta_1-32\beta_2-108\beta_3 -256\beta_4 =-256 \quad (2.22)$$

$$\alpha_1+32\alpha_2+243\alpha_3-5\beta_1-80\beta_2-405\beta_3 -1280\beta_4=-1024$$
$$\alpha_1+64\alpha_2+729\alpha_3-6\beta_1-192\beta_2 -1458\beta_3-6144\beta_4 =-4096$$
$$\alpha_1+128\alpha_2+2187\alpha_3-7\beta_1-448\beta_2 -5103\beta_3-28672\beta_4 =-16384$$

Using the maple software package to solve the system (2.22) we obtain the following:

$$\left\{\alpha_0=a,\alpha_1=\frac{1}{550}(2160a-1360),\alpha_2 =-\frac{1350}{550}(a+1),\alpha_3 =-\frac{1}{550}(1360a-2160),\beta_0 =-\frac{1}{550}(141a+9),\alpha_4 =1,\beta_1 =-\frac{1}{550}(1656a-456),\beta_2 =-\frac{2376}{550}(a-1),\beta_3 =-\frac{1}{550}(456a-1656),\beta_4 =\frac{1}{550}(9a+141)\right\}$$

On substituting the values into (2.2) we obtain:

$$y_{n+4} - \frac{1}{550}(1360a - 2160)y_{n+3} - \frac{1350}{550}(a+1)y_{n+2} + \frac{1}{550}(2160a - 1360)y_{n+1} + ay_n = \frac{h^3}{550}[(9a+141)f_{n+4} - (456a - 1656)f_{n+3} - 2376(a-1)f_{n+2} - (1656a - 456)f_{n+1} - (141a + 9)f_n] \quad (2.23)$$

Now on generating the methods by putting $a = 8, 19, 88$ and 92 into (2.23), we obtain the following:

at $a = 8$,

$$y_{n+4} - \frac{872}{55}y_{n+3} - \frac{143}{11}y_{n+2} + \frac{1592}{55}y_{n+1} + 8y_n = \frac{h^3}{550}[213f_{n+4} - 1992f_{n+3} - 16632f_{n+2} - 12792f_{n+1} - 1137f_n] \quad (2.24)$$

at $a = 19$,

$$y_{n+4} - \frac{2368}{55}y_{n+3} - \frac{540}{11}y_{n+2} + \frac{3968}{55}y_{n+1} + 19y_n = \frac{h^3}{550}[312f_{n+4} - 7008f_{n+3} - 42768f_{n+2} - 31008f_{n+1} - 2688f_n] \quad (2.25)$$

at $a = 88$,

$$y_{n+4} - \frac{11752}{55}y_{n+3} - \frac{2403}{11}y_{n+2} + \frac{18872}{55}y_{n+1} + 88y_n = \frac{h^3}{550}[933f_{n+4} - 38472f_{n+3} - 206712f_{n+2} - 145272f_{n+1} - 12417f_n] \quad (2.26)$$

at $a = 92$,

$$y_{n+4} - \frac{12296}{55}y_{n+3} - \frac{2511}{11}y_{n+2} + \frac{19736}{55}y_{n+1} + 92y_n = \frac{h^3}{550}[969f_{n+4} - 40296f_{n+3} - 216216f_{n+2} - 151896f_{n+1} - 12981f_n] \quad (2.27)$$

On using (2.24), (2.25), (2.26) and (2.27) as a block we have:

$$y_{n+4} - \frac{872}{55}y_{n+3} - \frac{143}{11}y_{n+2} + \frac{1592}{55}y_{n+1} + 8y_n = \frac{h^3}{550}[213f_{n+4} - 1992f_{n+3} - 16632f_{n+2} - 12792f_{n+1} - 1137f_n]$$

$$y_{n+4} - \frac{2368}{55}y_{n+3} - \frac{540}{11}y_{n+2} + \frac{3968}{55}y_{n+1} + 19y_n = \frac{h^3}{550}[312f_{n+4} - 7008f_{n+3} - 42768f_{n+2} - 31008f_{n+1} - 2688f_n] \quad (2.28)$$

$$y_{n+4} - \frac{11752}{55}y_{n+3} - \frac{2403}{11}y_{n+2} + \frac{18872}{55}y_{n+1} + 88y_n = \frac{h^3}{550}[933f_{n+4} - 38472f_{n+3} - 206712f_{n+2} - 145272f_{n+1} - 12417f_n]$$

$$y_{n+4} - \frac{12296}{55}y_{n+3} - \frac{2511}{11}y_{n+2} + \frac{19736}{55}y_{n+1} + 92y_n = \frac{h^3}{550}[969f_{n+4} - 40296f_{n+3} - 216216f_{n+2} - 151896f_{n+1} - 12981f_n]$$

RESULTS AND DISCUSSION

The block method for $k = 4$ has the following order and error constants. The method as a block is of order 5 and the error constant $\left[-\frac{27}{3080} \quad -\frac{60}{3080} \quad -\frac{267}{3080} \quad -\frac{279}{3080}\right]^T$.

Table 3.1: Order and Error Constant for $k = 4$.

Method	Order	Error Constant
$y_{n+4} - \frac{872}{55}y_{n+3} - \frac{143}{11}y_{n+2} + \frac{1592}{55}y_{n+1} + 8y_n = \frac{h^3}{550}[213f_{n+4} - 1992f_{n+3} - 16632f_{n+2} - 12792f_{n+1} - 1137f_n]$	5	$-\frac{27}{3080}$
$y_{n+4} - \frac{2368}{55}y_{n+3} - \frac{540}{11}y_{n+2} + \frac{3968}{55}y_{n+1} + 19y_n = \frac{h^3}{550}[312f_{n+4} - 7008f_{n+3} - 42768f_{n+2} - 31008f_{n+1} - 2688f_n]$	5	$-\frac{60}{3080}$
$y_{n+4} - \frac{11752}{55}y_{n+3} - \frac{2403}{11}y_{n+2} + \frac{18872}{55}y_{n+1} + 88y_n = \frac{h^3}{550}[933f_{n+4} - 38472f_{n+3} - 206712f_{n+2} - 145272f_{n+1} - 12417f_n]$	5	$-\frac{267}{3080}$
$y_{n+4} - \frac{12296}{55}y_{n+3} - \frac{2511}{11}y_{n+2} + \frac{19736}{55}y_{n+1} + 92y_n = \frac{h^3}{550}[969f_{n+4} - 40296f_{n+3} - 216216f_{n+2} - 151896f_{n+1} - 12981f_n]$	5	$-\frac{279}{3080}$

Convergence Analysis of the Block Formulae

The convergence of the block methods is determined using the approach in Fatunla [14-15], a block method can be defined as follows:
Let Y_m and F_m be vectors defined by:

$$Y_m = \begin{bmatrix} y_n \\ \vdots \\ y_{n+r-1} \end{bmatrix},\ F_m = \begin{bmatrix} f_n \\ \vdots \\ f_{n+r-1} \end{bmatrix},$$ respectively.

Then, a general k-block, r-point method is a finite difference equation of the form:

$$Y_m = \sum_{i=1}^{k} A_i Y_{m-i} + h^3 \sum_{i=0}^{k} B_i F_{m-i}$$

where all A_i's and B_i's are properly chosen, $r \times r$ matrix coefficients and $m = 0,1,2,\ldots$ represents the block number, $n = mr$ is the first step number in the m-th block and r the proposed block size.

Where according to Chu and Hamilton [10], the zero stability condition of the block is given as follows:

The block method is said to be zero-stable if the roots $R_j, j = 1(1)k$ of the first characteristics polynomial
$\rho(R) = \det\left[\sum_{i=0}^{k} A_i R^{k-i}\right] = 0, A_0 = -I,$
satisfies $|R_j| \le 1$. If one of the roots is $+1$, we call this root the principal root of $\rho(R)$.

The block differentiation formulae:

$$y_{n+4}-\frac{872}{55}y_{n+3}-\frac{143}{11}y_{n+2}+\frac{1592}{55}y_{n+1}+8y_n=\frac{h^3}{550}[213f_{n+4}-1992f_{n+3}-16632f_{n+2}-12792f_{n+1}-1137f_n]$$

$$y_{n+4}-\frac{2368}{55}y_{n+3}-\frac{540}{11}y_{n+2}+\frac{3968}{55}y_{n+1}+19y_n=\frac{h^3}{550}[312f_{n+4}-7008f_{n+3}-42768f_{n+2}-31008f_{n+1}-2688f_n]$$

$$y_{n+4}-\frac{11752}{55}y_{n+3}-\frac{2403}{11}y_{n+2}+\frac{18872}{55}y_{n+1}+88y_n=\frac{h^3}{550}[933f_{n+4}-38472f_{n+3}-206712f_{n+2}-145272f_{n+1}-12417f_n]$$

$$y_{n+4}-\frac{12296}{55}y_{n+3}-\frac{2511}{11}y_{n+2}+\frac{19736}{55}y_{n+1}+92y_n=\frac{h^3}{550}[969f_{n+4}-40296f_{n+3}-216216f_{n+2}-151896f_{n+1}-12981f_n]$$

is expressed as:

$$\begin{bmatrix}\frac{1592}{55} & -\frac{243}{11} & -\frac{872}{55} & 1\\ \frac{3968}{55} & -\frac{540}{11} & -\frac{2368}{55} & 1\\ \frac{18872}{55} & -\frac{2403}{11} & -\frac{11752}{55} & 1\\ \frac{19736}{55} & -\frac{2511}{11} & -\frac{12296}{55} & 1\end{bmatrix}\begin{bmatrix}y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & -8\\ 0 & 0 & 0 & -19\\ 0 & 0 & 0 & -88\\ 0 & 0 & 0 & -92\end{bmatrix}\begin{bmatrix}y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_n\end{bmatrix}+$$
$$h^3\begin{bmatrix}-\frac{12792}{550} & -\frac{166632}{550} & -\frac{1992}{550} & \frac{213}{550}\\ -\frac{31008}{550} & -\frac{42768}{550} & -\frac{7008}{550} & \frac{312}{550}\\ -\frac{145272}{550} & -\frac{206712}{550} & -\frac{38472}{550} & \frac{933}{550}\\ -\frac{151816}{550} & -\frac{216216}{550} & -\frac{40296}{550} & \frac{969}{550}\end{bmatrix}\begin{bmatrix}f_{n+1}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4}\end{bmatrix}+$$
$$h^3\begin{bmatrix}0 & 0 & 0 & -\frac{1137}{550}\\ 0 & 0 & 0 & -\frac{2688}{550}\\ 0 & 0 & 0 & -\frac{12417}{550}\\ 0 & 0 & 0 & -\frac{12981}{550}\end{bmatrix}\begin{bmatrix}f_{n-3}\\ f_{n-2}\\ f_{n-1}\\ f_n\end{bmatrix}$$

where

$$A=\begin{bmatrix}\frac{1592}{55} & -\frac{243}{11} & -\frac{872}{55} & 1\\ \frac{3968}{55} & -\frac{540}{11} & -\frac{2368}{55} & 1\\ \frac{18872}{55} & -\frac{2403}{11} & -\frac{11752}{55} & 1\\ \frac{19736}{55} & -\frac{2511}{11} & -\frac{12296}{55} & 1\end{bmatrix}; \quad B=\begin{bmatrix}0 & 0 & 0 & -8\\ 0 & 0 & 0 & -19\\ 0 & 0 & 0 & -88\\ 0 & 0 & 0 & -92\end{bmatrix};$$

$$C=\begin{bmatrix} -\frac{12792}{550} & -\frac{166632}{550} & -\frac{1992}{550} & \frac{213}{550} \\ -\frac{31008}{550} & -\frac{42768}{550} & -\frac{7008}{550} & \frac{312}{550} \\ -\frac{145272}{550} & -\frac{206712}{550} & -\frac{38472}{550} & \frac{933}{550} \\ -\frac{151816}{550} & -\frac{216216}{550} & -\frac{40296}{550} & \frac{969}{550} \end{bmatrix}$$ and

$$D=\begin{bmatrix} 0 & 0 & 0 & -\frac{1137}{550} \\ 0 & 0 & 0 & -\frac{2688}{550} \\ 0 & 0 & 0 & -\frac{12417}{550} \\ 0 & 0 & 0 & -\frac{12981}{550} \end{bmatrix}$$

It implies that

$$A^0=A^{-1}.A=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A^1=A^{-1}.B=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

for

$$A^{-1}=\begin{bmatrix} \frac{15125}{19832869112} & -\frac{15125}{19832869112} & -\frac{51425}{2479108639} & -\frac{8813335}{19832869112} \\ \frac{422675}{2479108639} & -\frac{422675}{2479108639} & -\frac{11496760}{2479108639} & \frac{11152350}{2479108639} \\ \frac{103688155}{2833267016} & -\frac{103688155}{2833267016} & \frac{1618650}{354158377} & -\frac{32689305}{2833267016} \\ \frac{43195115532}{27270195029} & -\frac{15924920503}{27270195029} & -\frac{72844640}{2479108639} & \frac{112409440}{2479108639} \end{bmatrix}$$

The first characteristics polynomial of the block method is given by $\rho(\lambda)=\det(\lambda A^0-A^1)$.

Substituting the A^0 and A^1 into the equation above gives:

$$\rho(\lambda)=det\begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}$$
$$\Rightarrow \lambda^3(\lambda-1)=0$$

This will yield $\lambda=0(thrice),\ \lambda=1$.

From the definition, the method is zero-stable and consistent since the order of the method $p=5>1$, which implies that the method is Convergent.

IMPLEMENTATION OF THE METHOD

The constructed block differentiation formulae for $4-step$ is tested on some initial value problems of order three differential equations. The results obtained from using these methods are compared with the exact solution.
The proposed methods were applied to the following third order differential equations.

Problem 4.1
Consider the initial value problem

$$y'''+5y''+7y'+3y=0, y(0)=1, y'(0)=0, y''(0)=-1, h=0.1,$$

whose exact solution is $y(x)=e^{-x}+xe^{-x}$. The results obtained are tabulated in the tables 4.1 to 4.3. This problem can also be found in Sagir [27].

Problem 4.2
Consider the initial value problem

$$y'''-y''+y'-y=0, y(0)=1, y'(0)=0, y''(0)=-1, h=0.01,$$

whose exact solution is $y(x)=cosx$. The results obtained are tabulated in the tables 4.4 to 4.6. This problem can also be found in Sagir [27].

Problem 4.1:
$$y'''+5y''+7y'+3y=0, y(0)=1, y'(0)=0, y''(0)=-1, h=0.1,$$

Exact solution: $y(x)=e^{-x}+xe^{-x}$.

Table 4.1: BBDF for $k = 4$.

X	Exact Value	Approximate Value	Error
0.1	0.9953211598	0.9953211590	2.62837 E-10
0.2	0.9824769037	0.9824769030	5.34269 E-10
0.3	0.9630636869	0.9630636860	8.00238 E-10
0..4	0.9384480644	0.9384480630	9.98623 E-10
0.5	0.9097959895	0.9097959850	1.12632 E-9
0.6	0.8780986178	0.8780986140	3.68204 E-9
0.7	0.8441950165	0.8441949720	4.36042 E-8
0.8	0.8087921354	0.8087920710	6.36142 E-8
0.9	0.7724823534	0.7724822650	8.78362 E-8
1.0	0.7357588824	0.73557587800	1.02061 E-7

Method: BBDF4

$$y_{n+4} - \frac{872}{55}y_{n+3} - \frac{143}{11}y_{n+2} + \frac{1592}{55}y_{n+1} + 8y_n = \frac{h^3}{550}[213f_{n+4} - 1992f_{n+3} - 16632f_{n+2} - 12792f_{n+1} - 1137f_n]$$

$$y_{n+4} - \frac{2368}{55}y_{n+3} - \frac{540}{11}y_{n+2} + \frac{3968}{55}y_{n+1} + 19y_n = \frac{h^3}{550}[312f_{n+4} - 7008f_{n+3} - 42768f_{n+2} - 31008f_{n+1} - 2688f_n]$$

$$y_{n+4} - \frac{11752}{55}y_{n+3} - \frac{2403}{11}y_{n+2} + \frac{18872}{55}y_{n+1} + 88y_n = \frac{h^3}{550}[933f_{n+4} - 38472f_{n+3} - 206712f_{n+2} - 145272f_{n+1} - 12417f_n]$$

$$y_{n+4} - \frac{12296}{55}y_{n+3} - \frac{2511}{11}y_{n+2} + \frac{19736}{55}y_{n+1} + 92y_n = \frac{h^3}{550}[969f_{n+4} - 40296f_{n+3} - 216216f_{n+2} - 151896f_{n+1} - 12981f_n]$$

Table 4.2: Comparison of BBDF4 with Sagir.

X	Exact Value	Approximate Value of BBDF4	Approximate Value of Sagir
0.1	0.9953211598	0.9953211590	0.9953212241
0.2	0.9824769037	0.9824769030	0.9824768765
0.3	0.9630636869	0.9630636860	0.9630636564
0..4	0.9384480644	0.9384480630	0.9384481542
0.5	0.9097959895	0.9097959850	0.9097955469
0.6	0.8780986178	0.8780986140	0.8780978452
0.7	0.8441950165	0.8441949720	0.8441930642
0.8	0.8087921354	0.8087920710	0.8087911080
0.9	0.7724823534	0.7724822650	0.7724810025
1.0	0.7357588824	0.7357587800	0.7357454120

Table 4.3: Error Comparison of BBDF4 with Sagir.

X	Exact Value	Error of BBDF4	Error of Sagir
0.1	0.9953211598	2.62837 E-10	6.4300 E-8
0.2	0.9824769037	5.34269 E-10	2.7200 E-8
0.3	0.9630636869	8.00238 E-10	3.0500 E-8
0..4	0.9384480644	9.98623 E-10	8.9800 E-8
0.5	0.9097959895	1.12632 E-9	4.4260 E-7
0.6	0.8780986178	3.68204 E-9	7.7260 E-7
0.7	0.8441950165	4.36042 E-8	1.9523 E-6
0.8	0.8087921354	6.36142 E-8	1.0274 E-6
0.9	0.7724823534	8.78362 E-8	1.3509 E-6
1.0	0.7357588824	1.02061 E-7	1.3470 E-5

Problem 4.2:

$y''' - y'' + y' - y = 0, y(0) = 1, y'(0) = 0, y''(0) = -1, h = 0.01,$

Exact solution: $y(x) = cosx$.

Table 4.4: BBDF for $k = 4$.

X	Exact Value	Approximate Value	Error
0.1	0.999999984	0.999999984	0.000000000
0.2	0.999999939	0.999999936	2.2923483 E-10
0.3	0.999999862	0.999999861	1.0023754 E-9
0..4	0.999999756	0.999999755	1.306075 E-9
0.5	0.999999619	0.999999611	7.303953325 E-9
0.6	0.999999451	0.999999442	9.214530850 E-9
0.7	0.999999253	0.999999238	1.531120045 E-8
0.8	0.999999025	0.999998995	3.000763250 E-8
0.9	0.999998766	0.999998709	5.682005862 E-8
1.0	0.999998476	0.999998197	2.794913288 E-8

Table 4.5: Comparison of BBDF4 with Sagir.

X	Exact Value	Approximate Value of BBDF4	Approximate Value of Sagir
0.1	0.9953211598	0.999999984	0.9999502003
0.2	0.9824769037	0.999999936	0.9998002023
0.3	0.9630636869	0.999999861	0.9995501702
0..4	0.9384480644	0.999999755	0.9992003588
0.5	0.9097959895	0.999999611	0.9987515643
0.6	0.8780986178	0.999999442	0.9982035679
0.7	0.8441950165	0.999999238	0.9975543456
0.8	0.8087921354	0.999998995	0.9968004658
0.9	0.7724823534	0.999998709	0.9959540620
1.0	0.7357588824	0.999998197	0.9950213456

Table 4.6: Error Comparison of BBDF4 with Sagir.

X	Exact Value	Error of BBDF4	Error of Sagir
0.1	0.9953211598	0.000000000	1.9990 E-7
0.2	0.9824769037	2.2923483 E-10	1.9560 E-7
0.3	0.9630636869	1.0023784 E-9	1.3651 E-7
0..4	0.9384480644	1.306075 E-9	2.5210 E-7
0.5	0.9097959895	7.303953325 E-9	1.3039 E-6
0.6	0.8780986178	9.214530820 E-9	3.0280 E-6
0.7	0.8441950165	1.531120045 E-8	3.3453 E-6
0.8	0.8087921354	3.000763250 E-8	1.2405 E-6
0.9	0.7724823534	5.682005862 E-8	1.3290 E-6
1.0	0.7357588824	2.794913288 E-7	1.7180 E-5

Test of Convergence of the Methods using Problem 4.1

From (2.19) which is the general form of BBDF4,

$\beta_k = \beta_3 = \frac{h^3}{33}(10 + a)$ and $\frac{\partial f}{\partial x} = 0 = L$

Since $h = 0.1$

$$\Rightarrow Lh|\beta_k| = 0 \times \frac{h^4}{33}(10 + a)$$
$$= 0 < 1$$

which implies that the method is convergent for every value of a.

Test of Convergence of the Methods using Problem 4.2

From (2.19) which is the general form of BBDF4,

$\beta_k = \beta_3 = \frac{h^3}{33}(10 + a)$ and $\frac{\partial f}{\partial x} = 0 = L$

Since $h = 0.1$

$\Rightarrow Lh|\beta_k| = 0 \times \frac{h^4}{33}(10 + a) = 0 < 1.$

which implies that the method is convergent for every value of a.

CONCLUSION

The constructed block method for $k = 4$ was used to solve problems of third-order differential equations. The consistency and zero-stability of the method was tested and hence the convergence of the method was established. In conclusion, the proposed block method is accurate as they produce results which compared favorably with the exact results and result obtained from other method.

Since backward differentiation methods are implicit, they have to be used in conjunction with a predictor, however iterating the corrector to convergence via fixed point iteration. We shall employ the corrector of backward difference method to convergence via fixed point iteration in our further studies.

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