Statistical Analysis of an Alternative von Bertalanffy Model for Tree Growth.

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ABSTRACT

This paper proposed an alternative von Bertalanffy nonlinear growth equation by introducing a growth rate stabilizing parameter θ using hyperbolic sine function. The ability of the integral solution of the alternative von Bertalanffy growth equation in model prediction was compared with the classical von Bertalanffy growth model, an approach which mimicked the natural variability of heights/diameter increment with respect to age and therefore provides a more realistic height/diameter predictions using the coefficient of determination (R^2), Mean Square Error (MSE) and Akaike Information Criterion (AIC) results. The "Kolmogorov-Smirnov" test, "Shapiro-Wilk" test and runs test were also used to test the compliance of the error term to the underlying assumptions. The mean function of top "height/diameter" over age using the two models under study predicted closely the observed values of top height/Dbh in the hyperbolic von Bertalanffy nonlinear growth models better than the classical von Bertalanffy growth model while the assumptions of the error term were maintained.

(Keywords: hight-diameter, von Bertalanffy growth model, hyberbolic, forest research, sustainability)

INTRODUCTION

Growth is one of the well-known features in biological organisms (Burkhart and Strub, 1974). Growth models describe the changing size of something over time. Forest growth models are very useful for forest managers and forestry researchers in many respects. A forest growth model aims to describe the dynamics of the forest closely and precisely enough to meet the needs of the forester or forestry researcher.

Karl Ludwig von Bertalanffy (September 19, 1901 – June 12, 1972) was an Austrian-born biologist known as one of the founders of general systems theory (GST). GST is an interdisciplinary practice that describes systems with interacting components, applicable to biology, cybernetics, and other fields. Bertalanffy proposed that the laws of thermodynamics applied to closed systems, but not necessarily to "open systems", such as living things. His mathematical model of an organism's growth over time, published in 1934, is still in use today (Rohner, 2013; Timilsina and Staudhammer, 2013). In its simplest version the so-called von Bertalanffy growth equation is expressed as a differential equation of length *(H)* over time *(t).*

Bertalanffy derived his equation from the assumptions, which he attributed to the fact that the rate of anabolism is proportional to the surface area of an organism (or to its mass raised to the power of 2/3), while catabolism is proportional to the organism's mass. These assumptions define what he calls "the first metabolic type".

Ricker (1979) questioned these assumptions considering them to be "fanciful speculations; nevertheless to honor the presumably original author of these speculations". Ricker refers to the Bertalanffy equation as the Pattern Growth Curve No. 2. He was concerned with the question," Why has individual growth a limit?" and proposed several answers that resemble structural and mechanical considerations put forth by Galileo in his 'Dialogues concerning two new sciences'. The model has the following differential form:

$$\frac{\partial H}{\partial t} = \frac{r}{b} H^{1-b}[K^b - H^b] \qquad (1)$$

A system can be formally defined as a set of elements also called components. A set of trees in a forest stand, producers and consumers in an economic system are examples of components. The elements (components) have certain characteristics or attributes and these attributes have numerical or logical values. Among the elements, relationships exist and consequently the elements are interacting. The state of a system is determined by the numerical or logical values of the attributes of the system elements. Experimenting on the state of a system with a model over time is termed simulation (Kansal et al., 2000; Li et. al., 2013; San Miguel et. al. 2004). Sustainable forest management relies to a large extent, measure on the predictions of the future conditions of individual stands which is achieved by predicting the increment from the current stand structure and updating the current values at each cycle of iteration using a functional growth model. Trees structural changes over time can be monitored and modeled under different cutting cycles, cutting intensities and optimal management policies can be arrived at based on the results of such simulation runs.

The process of developing a mathematical model is termed mathematical modeling. A model may help to explain a system and to study the effects of different components, and also to make predictions about behavior. A model may be deterministic or stochastic. A deterministic growth model gives an estimate of the expected growth of a system. Given the same initial conditions, a deterministic model will always predict the same result. A stochastic model attempts to illustrate natural variation by providing different predictions, each with a specific probability of occurrence.

Deterministic and stochastic models serve complementary purposes. In forestry, deterministic models are effective for determining the expected yield, and may be used to indicate the optimum stand condition. Stochastic models may indicate the reliability of these predictions, and the risks associated with any particular regime. Both deterministic and stochastic predictions can be obtained from some models. Although stochastic models can provide some useful information not available from deterministic models, most of the information needed for forest planning and managements can be provided efficiently also with the use of deterministic models.

Forest managers rely on growth and yield models to assess whether their short-term plans will meet long-term sustainability goals. Growth models assist forest researchers and managers in many ways. Some important uses include the ability to predict future yields and to help consider alternative cultivation practices.

Models provide an efficient way to prepare resource forecasts, but a more important role may be their ability to explore management options and silvicultural alternatives (Ek, Birdsall, and Spear, 1984; Oyamakin and Chukwu, 2014). Growth models provide a reliable way to examine silvicultural and harvesting options, to determine the sustainable timber yield, and examine the impacts of forest management and harvesting on other values of the forest. Forest managers may require information on the present status of the resource (e.g. numbers of trees by species and sizes for selected strata), forecasts of the nature and timing of future harvests, and estimates of the maximum sustainable harvest. Forest simulation models or forest growth models are very useful for forest managers and forestry researchers in many respects. A forest growth model aims to describe the dynamics of the forest closely and precisely enough to meet the needs of the forester or forestry researcher.

Growth models assist forest researchers and managers in many ways. Some important uses include the ability to predict future yields and to explore silvicultural options (Oyamakin et al., 2013). Models provide an efficient way to prepare resource forecasts, but a more important role may be their ability to explore management options and silvicultural alternatives. For example, foresters may wish to know the long-term effect on both the forest and on future harvests, of a particular silvicultural decision, such as changing the cutting limits for harvesting. With a growth model, they can examine the likely outcomes (Myers, 1996) both with the intended and alternative cutting limits, and can make their decision objectively. The process of developing a growth model may also offer interesting new insights into stand dynamics.

The total height (Ht) of a tree is important for computing and estimating tree volume, stand characteristics and features through site index, but accurate measurement of this variable is time consuming. As a result, foresters often choose to measure only a few trees' heights and estimate the remaining heights with height-diameter equations. Foresters can also use height-diameter equations to indirectly estimate height growth by applying the equations to a sequence of diameters that were either measured directly in a continuous inventory or predicted indirectly by a diameter-growth equation (Zeide, 1993). The diameter-growth prediction method is very useful in modeling growth and yield of trees due to lack of approximations in measuring the diameter of trees. Curtis (1967) investigated several equations for Douglas-Fir that included tree diameter outside bark at breast height (DBH) as an explanatory variable.

In this paper, an alternative nonlinear growth model called the hyperbolic von Bertalanffy growth model was introduced and compared with the existing classical von Bertalanffy model using Pine height/diameter data. The data was obtained from the growth and yield section of the sustainable forest management department, Forestry Research Institute of Nigeria, Ibadan.

BACKGROUND INFORMATION AND METHODS

In this section, we discussed the background information and methods through which tree growth can be model. We also, discussed the hyperbolic sine function as used by Tabatabai (2005). Consider a nonlinear model:

$$H_i = f(D_i, \boldsymbol{B}) + \varepsilon_i \qquad (2)$$

$i = 1,2, \dots, n$, where H is the response variable, D is the independent variable, **B** is the vector of the parameters β_j to be estimated $(\beta_1, \beta_2 \dots \dots, \beta_p)$,$\varepsilon_i$ is a random error term , p is the number of unknown parameters, n is the number of observation. The estimator of β_j's are found by minimizing the sum of squares residual (SS_{Rss}) function:

$$SS_{Rss} = \sum_{i=1}^{n}[H_i - f(D_i, B)]^2 \qquad (3)$$

under the assumption that ε_i are normal and independent with mean zero and common variance σ^2. Since H_i and D_i are fixed observations, the sum of squares residual is a function of **B**, these normal equations take the form of

$$\sum_{i=1}^{n}\{H_i - f(D_i, \boldsymbol{B})\}\left[\frac{\partial f(D_i, B)}{\partial \beta_j}\right] = 0 \qquad (4)$$

"for $j = 1,2, \dots, p$." When the model is nonlinear in the parameters so are the normal equations consequently, for the nonlinear model consider Table 2, it is impossible to obtain the closed solution of the least squares estimate of the parameter by solving the p normal equations describe in Equation 3. Hence an iterative method must be employed to minimize the SS_{Res} (Draper and Smith 1981, Ratkowsky 1983, Marquardt 1963, Seber and Wild 1989, Fekedulegn 1996).

The hyperbolic functions have similar names to the trigonometric functions, but they are defined in terms of the exponential function. The three main types of hyperbolic functions, and the sketch of their graphs are giving below.

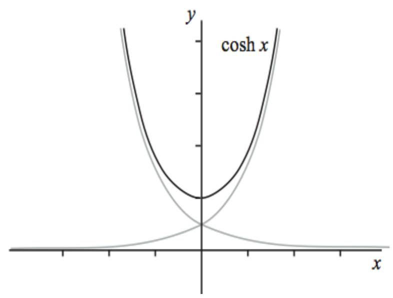


Figure 1: Cosh Function.

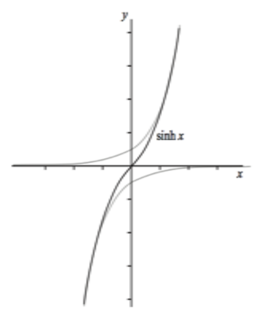


Figure 2: Sinh Function.

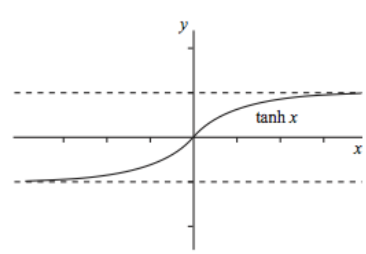


Figure 3: Tanh Function.

For completeness we follow, the hyperbolic sine function and its inverse provide an alternative method for evaluating:

$$\int \frac{1}{\sqrt{1+x^2}}dx \qquad (5)$$

If we make the substitution, then:

$$\sqrt{1+x^2} = \sqrt{1+sinh^2(u)} = \sqrt{cosh^2(u)} = \cosh(u) \qquad (6)$$

where the second equality follows from the identity $\cosh^2(u) - \sinh^2(u) = 1$ and the last equality from the fact that cosh(u) > 0 for all u. Hence:

$$\int \frac{1}{\sqrt{1+x^2}}dx = \int \frac{\cosh(u)}{\cosh(u)}du = \int du = u + c = sinh^{-1}(x) + c \qquad (7)$$

The following proposition is a consequence of the integral in Equation (7) i.e.,

$$\frac{d}{dx} sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}} \qquad (8)$$

Also, using the substitution x = tan (u), $-\frac{\pi}{2} < u < \frac{\pi}{2}$, that:

$$\int \frac{1}{\sqrt{1+x^2}}dx = log\left|x + \sqrt{1+x^2}\right| + c \qquad (9)$$

Since two anti-derivatives of a function can differ at most by a constant, there must exist a constant k such that:

$$sinh^{-1}(x) = log\left|x + \sqrt{1+x^2}\right| + k \qquad (10)$$

for all x. Evaluating both sides of this equality at x = 0, we have:

$$0 = sinh^{-1}(0) = \log(1) + k = k \qquad (11)$$

Thus k = 0 and

$$sinh^{-1}(x) = log\left|x + \sqrt{1+x^2}\right| \qquad (12)$$

for all x. Since the hyperbolic sine function is defined in terms of the exponential function, we should not find it surprising that the inverse hyperbolic sine function may be expressed in terms of the natural logarithm function. Hyperbolic models have been used before in growth modeling. The terminology of "hyperbolastic" was used in Tabatabai et al. (2005).

HYPERBOLASTIC VON BERTALANFFY GROWTH MODEL (HVBGM)

$$\frac{dH}{dt} = H^{1-b}[K^b - H^b]\left[\frac{r}{b} + \frac{\theta}{\sqrt{1+t^2}}\right] \qquad (13)$$

Separating variables we have that:

$$\frac{dH}{[K^b H^{1-b} - H]} = \left[\frac{r}{b} + \frac{\theta}{\sqrt{1+t^2}}\right]dt \qquad (14)$$

Divide the LHS by H^{1-b} to have;

$$\frac{H^{b-1}}{[K^b - H^b]}dH = \left[\frac{r}{b} + \frac{\theta}{\sqrt{1+t^2}}\right]dt \qquad (15)$$

The H integral looks difficult hence,

Let $m = K^b - H^b$ such that

$$dm = -bH^{b-1}dH$$

$$dH = \frac{dm}{-bH^{b-1}}$$

Substituting we have that:

$$-\frac{1}{b}\frac{dm}{m} = \left[\frac{r}{b} + \frac{\theta}{\sqrt{1+t^2}}\right]dt \qquad (16)$$

On integration we have that:

$$\ln m^{-\frac{1}{b}} = \frac{rt}{b} + \theta arcsinh(t) + C_1 \qquad (17)$$

Substituting back the value of m and taking the exponential of both sides gives:

$$(K^b - H^b)^{-\frac{1}{b}} = Ae^{\frac{rt}{b} + \theta arcsinh(t)} \qquad (18)$$

Finally, solving for H gives the hyperbolic von Bertalanffy growth model:

$$H = \left[K^b - Ae^{-\frac{rt}{b} - \theta arcsinh(t)}\right]^{\frac{1}{b}} \qquad (19)$$

Therefore, we shall apply the two models below on Age-Height and Age-Diameter of pines (*Pinus caribaea*) growth:

(1) $H = \left[K^b - Ae^{-\frac{rt}{b} - \theta arcsinh(t)}\right]^{\frac{1}{b}} + \varepsilon$, and $D = \left[K^b - Ae^{-\frac{rt}{b} - \theta arcsinh(t)}\right]^{\frac{1}{b}} + \varepsilon$

(2) $H = [K^b - Ae^{-rt}]^{1/b} + \varepsilon$, and $D = [K^b - Ae^{-rt}]^{1/b} + \varepsilon$

RESULTS AND DISCUSSION

In this section we discuss the result obtained by comparing the proposed model over its source model. This was achieved using general fitness in term of model selection criteria, graphs and tables.

Tables 1-4 show the estimated parameter for exponential and hyperbolic exponential growth model while Table 5 shows their respective coefficient of determination (R^2), MSE, and AIC for age-height/age-diameter models.

Table 1: Height Parameter Estimates using Von-Bertalanffy Growth Model.

Parameter	Estimate
A	9.711
r	0.012
k	1525.444
b	0.333

Table 2: Height Parameter Estimates using Hyperbolic von Bertalanffy Growth Model.

Parameter	Estimate
A	1.001
r	1.539E-009
k	308.644
b	0.00001
θ	0.0001

Table 3: Diameter Parameter Estimates using von Bertalanffy Growth Model.

Parameter	Estimate
A	18.250
r	0.010
k	813.999
b	0.456

Table 4: Diameter Parameter Estimates using Hyperbolic von Bertalanffy Growth Model.

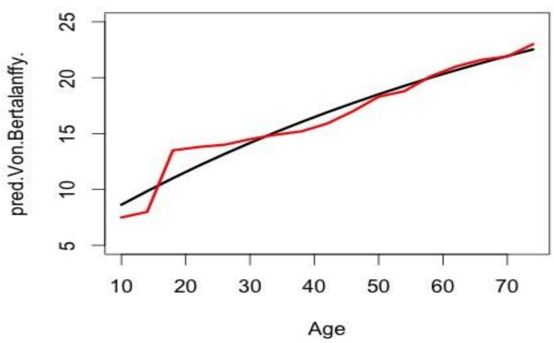
Parameter	Estimate
A	8.683
r	0.041
k	26.115
b	1.067
θ	-0.415

Table 5: Summary of Model Selection Criteria Computed for the Proposed and Source Models.

Models	SSE	N	K	R^2	MSE	AIC
SOURCE(Ht)	17.548	17	4	94.60%	1.35	8.539352867
PROPOSED(Ht)	14.843	17	5	95.40%	1.237	7.693355503
SOURCE(Dbh)	5.88	17	4	98.70%	0.452	-10.0481619
PROPOSED(Dbh)	3.053	17	5	99.30%	0.254	-19.19050672

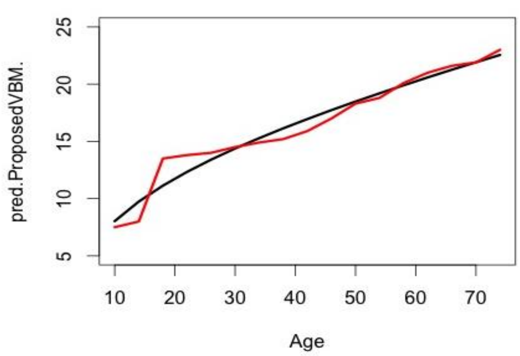
SSE- RESIDUAL SUM OF SQUARES, N- NO OF OBSERVATIONS, MSE- MEAN SQUARE ERROR, AIC- AKAIKE INFORMATION CRITERION

Also, the predicted and observed height and diameter were plotted to show the relationship and how best the models predicted the observed data on height and diameter of pines. This is also shown in the figure below:



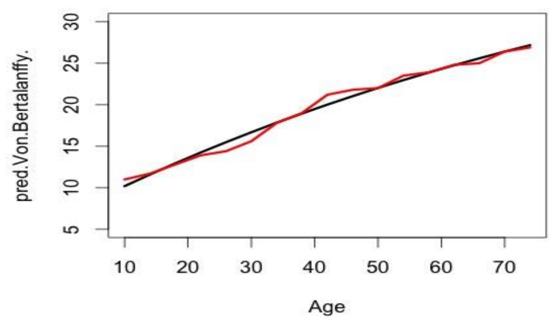
BLACK = Observed data & RED = Predicted

Figure 4: Observed Height (Data) against Predicted Height (von Bertalanffy Growth Model).



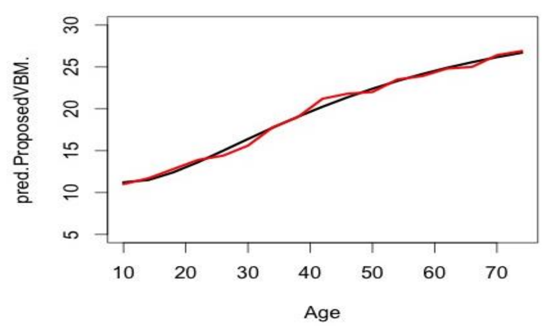
BLACK = Observed data & RED = Predicted

Figure 5: Observed Height (Data) against Predicted Height (Hyperbolic von Bertalanffy Growth Model).



BLACK = Observed data & RED = Predicted

Figure 6: Observed Diameter (Data) against Predicted Diameter (von Bertalanffy Growth Model).



BLACK = Observed data & RED = Predicted

Figure 7: Observed Diameter (Data) against Predicted Diameter (Hyperbolic von Bertalanffy Growth Model).

Testing for Independence of Errors (Run Test) and Normality of Error (Shapiro-Wilk Test)

Two assumptions made in the models are:

- Errors are independent
- Errors are normally distributed.

These assumptions were verified by examining the residuals. If the fitted models are correct, residuals should exhibit tendencies that tend to confirm or at least should not exhibit a denial of the assumptions. Hence, we tested the following hypotheses stated below:

H_0: Errors are independent (Using Runs Test)
H_1: Errors are not independent

And

H_0: Errors are normally distributed (Using Shapiro-Wilk test)
H_1: Errors are not normally distributed

Let *m* be the number of pluses and *n* be the number of minuses in the series of residuals. The test is based on the number of runs(r), where a run is defined as a sequence of symbols of one kind separated by symbols of another kind. A good large sample approximation to the sampling distribution of the number of runs is the normal distribution with mean;

$$Mean = \frac{2mn}{m+n} + 1 \qquad (20)$$

and,

$$Variance(\sigma^2) = \frac{2mn(2mn - m - n)}{(m+n)^2(m+n-1)} \qquad (21)$$

Therefore, for large samples like ours the required test statistic is:

$$Z = \frac{(r + h - \mu)}{\sigma} \sim N(0,1) \qquad (22)$$

where,

$$h = \begin{cases} 0.5, if\ r < \mu \\ -0.5, r > \mu \end{cases}$$

Also, the required test statistic for the test of normality (Shapiro-Wilk test) is given by:

$$W = \frac{S^2}{b} \qquad (23)$$

Where;

$$S^2 = \sum a(k)\{x_{n+1-k} - x_{(k)}\} \qquad (24)$$

and,

$$b = \sum (x_i - \bar{x})^2 \qquad (25)$$

In the above, the parameter k takes the values; $x_{(k)}$ is the k^{th} order statistic of the set of residuals and the values of coefficient a(k) for different values of n and k are given in the Shapiro-Wilk table. H_0 is rejected at level α i.e. W is less than the tabulated value.

Tables 6 and 7 below do not reject the null hypothesis stated which suggested that normality and independent assumption hold except the error component of the classical Von Bertalanffy growth model which is significant at 10% when applied to diameter-Age data of pines (*Pinus caribaea*).

Table 6: Result of the Test of Independence of Residuals using Run Test.

Residual	Test Value	No of Runs	Z	Asmp. Sig.(2 tailed)
VB. Height	-0.0006	6	-1.494	0.135^{ns}
VB. Diameter	0.0000	8	-0.488	0.626^{ns}
HVB. Height	0.0000	6	-1.494	0.135^{ns}
HVB. Diameter	-0.0006	10	-0.015	0.988^{ns}

NS = not significant

Table 7: Result of the Test of Normality of Residuals using K-S & S-W Tests.

Residual	Kolmogorov-Smirnov		Shapiro-Wilk	
	Statistic	Asmp. Sig.	Statistic	Asmp. Sig.
VB. Height	0.150	*0.200^{ns}*	0.961	*0.649^{ns}*
VB. Diameter	0.193	*0.094**	0.945	*0.386^{ns}*
HVB. Height	0.149	*0.200^{ns}*	0.958	*0.592^{ns}*
HVB. Diameter	0.131	*0.200^{ns}*	0.973	*0.863^{ns}*

** Significant at 10%, and NS = not significant*

CONCLUSION

In conclusion, our proposed model shows promising results when applied to data on age, diameter, and height of pines. Our proposed model fitted the data with smaller AIC, MSE, and higher prediction accuracy than their source model. Hence, choosing a flexible and highly accurate predictive model such as ours can significantly improve the outcome of a study because the accuracy of a model is what determines its utility. We therefore recommend usage of our proposed models to the scientific community and practitioners and urge comparison of them with classical models before decisions on model selection are made. We have succeeded in introducing a new growth model using the hyperbolic sine function. The mean function of top height and Dbh over age using the Hyperbolic von Bertallanfy growth model predicted closely the observed values of top height and diameter of pines better than the source model.

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