Correlation of Adomian Decomposition and Finite Difference Methods in Solving Nonhomogeneous Boundary Value Problem.

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ABSTRACT

In this paper, we show that the result of Adomian Decomposition Method (FDM) and Finite Difference Method (FDM) are in agreement with each other, and the result of FDM gets better as the step size reduces. The two methods are applied to a single linear, inhomogeneous equation with different step size. The result shows that the methods are reliable, accurate and converges rapidly to the desired result. A smart 3D cubic spline-fit of the three data set using Maple shows the same result.

(Keywords: Adomian decomposition method, finite difference method, second order non-homogenous differential equation)

INTRODUCTION

It is often difficult to find a closed form solution to many differential equations especially the nonhomogeneous linear ordinary and partial differential equations. Nonetheless, there are many methods for finding approximate solutions. ADM is a recent, ingenious method for solving nonlinear functional equation of various kinds. It has been used in to solve a wide class of stochastic and deterministic problems involving nonlinear and integral equations.

ADM provides a solution as an infinite series in which each term can be determined, Some of the literatures on ADM include [2,3,4,5] In FDM, the problem is discretized and the solution given at interval, some existing literatures on FDM include [1,5].

We shall consider equation of the form:

$$y'' = f(x,y), y(a) = y_0 \; y(b) = y_n, x \in [a,b] \quad \textbf{(1)}$$

THE ADOMIAN DECOMPOSITION METHOD (ADM)

Following the analysis by Adomian [3] (1) exist and satisfies the Lipchitz condition and is written in operator form as:

$$Ly = f(x,y) \quad \textbf{(2)}$$

where

$$L = \frac{d^2}{dx^2}$$

The inverse operator:

$$L^{-1} = \int_0^x \int_0^x (.)\, dxdx \quad \textbf{(3)}$$

is a twofold integral. Taking L^{-1} on both sides of (2) and imposing and boundary condition we obtain:

$$y(x) = y_0 + y_1 x + L^{-1}(f(x,y)) \quad \textbf{(4)}$$

y(x) is given by infinite series of components:

$$y(x) = \sum_{n=0}^{\infty} A_n \quad \textbf{(5)}$$

and the nonlinear function f(x,y) by an infinite series of the form:

$$f(x,y) = \sum_{n=0}^{\infty} A_n \quad \textbf{(6)}$$

where the component $y_n(x)$ of the solution y(x) will be determined recurrently and the Adomian polynomial A_n is given as:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[f(\sum_{i=0}^{\infty}\lambda^i y_i)]_{\lambda=0} \quad \textbf{(7)}$$

n = 0, 1, 2, ...

on substituting **(5)** and **(6)** in **(4),** we obtain:

$$\sum_{n=0}^{\infty} y_n(x) = y_0 + y_1 x + L^{-1}(\sum_{n=0}^{\infty} A_n) \quad \textbf{(8)}$$

Each term of **(5)** is given by the recurrent relation:

$$\begin{aligned} y_0(x) &= y_0 + y_1 x \\ y_{n+1}(x) &= L^{-1}A_n, \quad n \geq 0 \end{aligned} \quad \textbf{(9)}$$

The solution of **(1)** will be approximated by series of the form:

$$\phi_k(x) = \sum_{n=0}^{k-1} y_n(x) \quad \textbf{(10)}$$

FINITE DIFFERENCE METHOD (FDM)

FDM are the implicit or the explicit relations between the derivatives and the function values at the adjacent mesh points. The mesh points on [a,b] may be defined as:

$$x_j = a + jh,\ j = 1, 2, ..., N+1$$

where $x_0 = a, x_{n+1} = b$ and $h = \frac{b-a}{N+1}$

For a boundary value problem, FDM is applied by replacing the differential equation at each mesh point by difference equations. Incorporating the boundary conditions in the difference equation and solving the resulting system of algebraic equations. This gives the approximate numerical solution of the boundary value problem **(1)**.

The exact value of y(x) at x_j is denoted by Y_j and its approximate value by y_j using Taylors series:

$$y'(x_j) = \frac{y(x_{j+1}) + y(x_{j-1})}{2h} - \frac{h^2}{6} y'''(\eta_j) \quad \textbf{(11)}$$

where, $x_{j-1} \triangleleft \eta_j \triangleleft x_{j+1}$ and

$$y''(x_j) = \frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{h^2} - \frac{h^2}{12} y^{iv}(\lambda_j) \quad \textbf{(12)}$$

where $x_{j-1} \triangleleft \lambda_j \triangleleft x_{j+1}$.

Assuming the continuity of $y'''(\eta_j)$ and $y^{iv}(\lambda_j)$ and neglecting $0(h^2)$ terms in **(11)** and **(12)** and substituting in **(1)** gives a system which is given in matrix notation as:

$$Ay = b \quad \textbf{(13)}$$

where b, y are n x 1 matrices and A is a tridiagonal square matrix. The solution for y in **(13)** gives the finite difference approximation of **(1)**.

APPLICATION AND RESULT

Example

Consider

$$y'' = x + y,\ y(0) = 1,\ y(5) = 143.4131591 \quad \textbf{(14)}$$

(i) Applying ADM to (14), we obtain:

$$Ly = x + y \quad \textbf{(15)}$$

operating L^{-1} on both sides of (15) and imposing the initial condition, we obtain:

$$y(x) = 1 + L^{-1}(x) + L^{-1}(y) \quad \textbf{(16)}$$

The ADM introduces a recursive relation:

$$y_0 = A_0$$

$$y_1 = L^{-1}(A_0)$$

In this order we obtain:

$$y_n = y_n(x) = \frac{x^{2n}}{(2n)!} + \frac{x^{3+2n}}{(3+2n)!}$$

Hence,

$$\phi_{10}(x) = \sum_{n=0}^{9} y_n(x)$$

See Table 1 for results.

TABLE 1
At h = 0.5

x	Exact	ADM	FDM
0.00	1.00000000	1.00000000	1.00000000
0.50	1.14872127	1.14872134	1.19396704
1.00	1.71828183	1.71828175	1.81142584
1.50	2.98168987	2.98168921	3.13174110
2.00	5.38905610	5.38905621	5.60999162
2.50	9.68249396	9.68249416	9.99074005
3.00	17.0855369	17.0855370	17.4941735
3.50	29.6154520	29.6154556	30.1211503
4.00	50.5981500	50.5981522	51.1534147
4.50	85.5171313	85.5171356	85.9740328
5.00	143.413162	143.413162	143.413159

(ii) By FDM, the discretized scheme for **(14)** is given as:

$$y_{n+1} - 2y_n + y_{n-1} - h^2 y_n - h^2(x_0 + nh) = 0 \qquad \textbf{(17)}$$

With step size of 0.5 **(17)** becomes:

$$y_{n+1} - 2y_n + y_{n-1} - 0.25y_n - 0.125n = 0 \qquad \textbf{(18)}$$

The solution of **(18)** is of the form **(13)** which is a linear nonhomogeneous system with:

$$A = \begin{bmatrix} p\,1\,0\,0\,0\,0\,0\,0\,0 \\ 1\,p\,1\,0\,0\,0\,0\,0\,0 \\ 0\,1\,p\,1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,p\,1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,p\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1\,p\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1\,p\,1\,0 \\ 0\,0\,0\,0\,0\,0\,1\,p\,1 \\ 0\,0\,0\,0\,0\,0\,0\,1\,p \end{bmatrix}$$

where p = -2.25, and

$$b = \begin{bmatrix} -0.875 \\ 0.250 \\ 0.375 \\ 0.500 \\ 0.625 \\ 0.750 \\ 0.875 \\ 1.000 \\ -142.2881591 \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \end{bmatrix}$$

From **(13)** $y = A^{-1} b$

The augmented coefficient 9x10 matrix is:

$$[A \vdots b]_{9x10} \qquad \textbf{(19)}$$

In echelon form Maple 14 reduces (**19)** to the form:

$$[I \vdots C]_{9x10}$$

Where I is and identify 9x9 matrix and C = y_j, j = 1, 2, ...,9, is 9x1 matrix given as:

$$y = \begin{bmatrix} 1.19396704 \\ 1.81142584 \\ 3.13174170 \\ 5.60999162 \\ 9.99074005 \\ 17.4941735 \\ 30.1211503 \\ 51.1534147 \\ 85.9740328 \end{bmatrix}$$

Table 1 compares the result obtained using ADM and FDM with exact solution of **(14).**

<u>Example</u>

$$y'' = x + y,\ y(0) = 1,\ y(1) = 1.71828210 \qquad \textbf{(20)}$$

By FDM, the discretized equivalent of **(20)** with h = 0.1 is given as:

$$y_{n+1} - 2.01y_n + y_{n-1} - 0.001n = 0 \qquad \textbf{(21)}$$

The system (21) has p = -2.01

$$b = \begin{bmatrix} -0.990 \\ 0.002 \\ 0.003 \\ 0.004 \\ 0.005 \\ 0.006 \\ 0.007 \\ 0.008 \\ -1.708281828 \end{bmatrix} \quad y_j = \begin{bmatrix} 1.00513610 \\ 1.02132356 \\ 1.04972425 \\ 1.09916222 \\ 1.14843635 \\ 1.22173487 \\ 1.31325074 \\ 1.42489912 \\ 1.55879649 \end{bmatrix}$$

Table 2 gives a similar result obtained using ADM, FDM and exact solution at h = 0.1.

CUBIC SPLINE FIT OF TABLE 1 DATA

Taking the data point at h = 0.5 as shown in Table 1, the cubic spline fit of Exact solution, ADM and FDM are shown in 3D plot using Maple in Figure1, Figure 2, and Figure 3, respectively.

The cubic spline fit is such that (i) f(x) is a linear polynomial outside the interval [0,5], (ii) f(x) is cubic polynomial in each of the subinterval and (iii) $f'(x)$ and $f''(x)$ are continuous at each point. Since we took equally-spaced valued of x $(x_{i+1} - x_i)$ = h, we can write

$$f''(x) = \frac{1}{h}\left[(x_{i+1} - x)f''(x_i) + (x - x_i)f''(x_{i+1})\right]$$

integrating twice, we obtain:

$$f(x) = \frac{1}{h}\left[\frac{(x_i - x)^3}{3!}f''(x_i) + \frac{(x - x_i)^3}{3!}f''(x_{i+1})\right] + \alpha_{1i}(x_{i+1} - x) + \alpha_{2i}(x - x_i) \qquad \textbf{(22)}$$

α_{1i} and α_{2i}k are determined by substituting the values of y = f(x) at x_{i+1} and x_i. Thus

$$\alpha_{1i} = \frac{1}{h}\left[y_i - \frac{h^2}{3!}f''(x_i)\right]$$

and

$$\alpha_{2i} = \frac{1}{h}\left[y_{i+1} - \frac{h^2}{3!}f''(x_{i+1})\right]$$

TABLE 2
At h = 0.1

x	Exact	ADM	FDM
0.00	1.00000000	1.00000000	1.00000000
0.10	1.00507092	1.00517094	1.00513610
0.20	1.02140276	1.02140284	1.02132356
0.30	1.04985881	1.04985881	1.04972425
0.40	1.09182470	1.09182465	1.09916221
0.50	1.14872128	1.14872134	1.14843635
0.60	1.22211880	1.22211874	1.22173487
0.70	1.31375271	1.31375277	1.31325074
0.80	1.42554093	1.42554104	1.42489912
0.90	1.44960311	1.44960333	1.55879649
1.00	1.71828183	1.71828108	1.71828210

substituting α_{1i} and α_{2i} in (22) and writing $f''(x_i) = M_{i,}$ we obtain:

$$f(x) = \frac{(x_{i+1}-x)^3}{6h} M_i + \frac{(x-x_i)^3}{6h} M_{i+1}$$

$$+ \frac{(x_{i+1}-x)^3}{h}\left(y_i - \frac{h^2}{6}\right) M_i$$

$$+ \frac{x-x_i}{h}\left(y_{i+1} - \frac{h^2}{6} M_{i+1}\right) \qquad \textbf{(23)}$$

$$\therefore\ f'(x) = -\frac{(x_{i+1}-x)^2}{2h} M_i + \frac{(x-x_i)^2}{2h} M_{i+1}$$
$$-\frac{h}{6}\left(M_{i+1} - M_i\right) + \frac{1}{h}\left(y_{i+1} - y_i\right)$$

To impose the condition of continuity of $f'(x)$, we set:

$f'(x-\varepsilon) = f'(x+\varepsilon)$ as $\varepsilon \to \infty$

$$\therefore\ \frac{h}{6}\left(2M_i + M_{i-1}\right) + \frac{1}{h}\left(y_i - y_{i-1}\right)$$

$$= \frac{h}{6}\left(2M_i + M_{i+1}\right) + \frac{1}{h}\left(y_{i+1} + y_i\right)$$

or

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) \quad \textbf{(24)}$$

i = 1 to n – 1 (in this article i = 1 to 9. For $x \lhd x_0$ and $x \rhd x_{n,}$ we have:

$$M_o = 0, \quad M_n = 0 \qquad \textbf{(25)}$$

(24) and (25) gives (n+1) equations in (n+1) unknowns. Solving for M_i and substituting in (23) we get the concerned cubic spline. In this article this was taken care of by Maple.

CONCLUSION

Table 1 and Table 2 have shown that ADM and FDM are not in total disagreement with each other. FDM gives result closer to that of exact solution as the step size decreases.

Also, the cubic spline fit of the three data set on the same interval has continuous derivatives which makes the 3D graph of the solution function appear smooth. Figures 1 - 3 have the same shapes and sizes that ensured curvature continuity. This has further illustrated that although the data set of ADM and FDM appeared slightly different from the analytical solution, their cubic spline fit are the same.

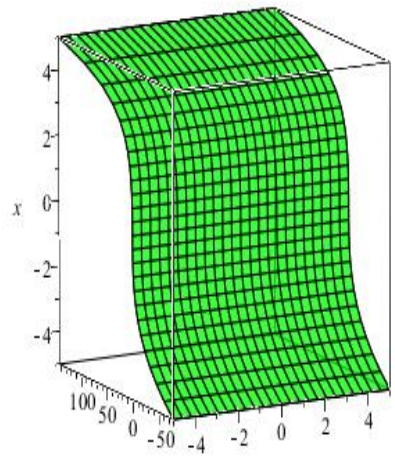

Figure 1: 3D Cubic Spline Fit of Exact Solution.

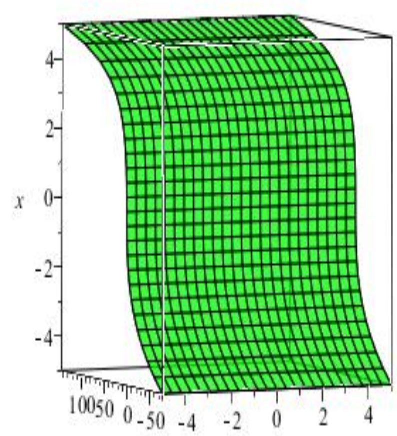

Figure 2: 3D Cubic Spline Fit of ADM.

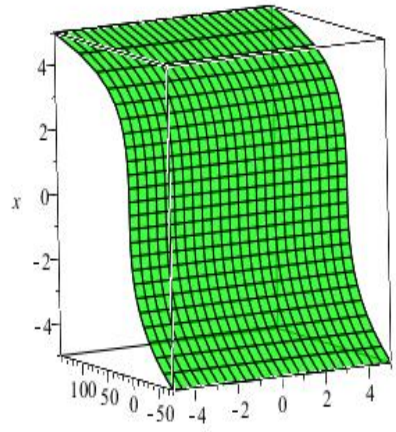

Figure 3: 3D Cubic Spline Fit of FDM.

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