Misrepresentations (and Corrections) of Routh's Stability Criterion.

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ABSTRACT

Routh's Stability Criterion for linear systems, as presented by several authors, is stated. Through research, it is discovered that an author stated the criterion wrongly. This results in an ambiguity and contradiction about the criterion. An illustrative mathematical problem reveals the ambiguity very clearly. A proper expression for Routh's Stability Criterion is suggested.

(Keywords: Routh's stability, criterion, control, systems, Ogata, Hurwitz)

INTRODUCTION

Routh's stability criterion, developed by E.J. Routh in 1905 (Wikipedia, 2014), is one of the widely-applied linear systems stability criteria. Other stability criteria that are similarly applied to determine linear systems stability are Hurwitz, continued fraction, Nyquist, and Root Locus. They all make use of the characteristic equation of the transfer function of a closed-loop control system, directly or indirectly.

Another criterion that is applied to determine the stability of control systems (linear and non-linear), is Lyapunov's stability criterion (Drazin, 2013). Lyapunov's stability criterion does not make use of characteristic equation of the transfer function of a closed-loop control system to determine the stability of the system. Rather, it uses differential equations about a point of interest to investigate the stability of the system about such point. The focus of this work is however on Routh's stability criterion.

Routh's stability criterion is particularly interesting as it utilizes a mathematical array obtained from the characteristic equation and thereafter only concerns itself with the signs of the coefficients of the first column of the array to determine the stability of the control system under study. This stability criterion is quite an old one and has stood the test of time. However, through research, a recent discovery by the author of this paper is that the criterion is wrongly presented by some authors; a presentation which has totally negated the very essence for which it was developed by Routh himself in 1905. The following presentations shall confirm the claim.

ROUTH'S STABILITY CRITERION

Routh's stability criterion, as earlier mentioned, makes use of the generic expression of the characteristic equation obtained from the transfer function of a closed-loop control system. Such equation is generally expressed as:

$$a_o s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \ldots + a_{n-1} s^1 + a_n = 0 \quad \textbf{(1)}$$

Routh, who researched and came up with this criterion, presented a mathematical array which he developed from equation (1) above (and called it Routh array), as follows:

s^n:	a_o	a_2	a_4	a_6
s^{n-1}:	a_1	a_3	a_5	a_7
s^{n-2}:	b_1	b_2	b_3	b_4
s^{n-3}:	c_1	c_2	c_3	c_4
s^{n-4}:	d_1	d_2	d_3	d_4
.	.			
.	.			
.	.			
.	.			
.	.			
.	.			
S^2:	e_1	e_2		
S^1:	f_1			
S^o:	g_1			

(2)

(Nagrath *et al*., 2009, 2nd ed.).

where:

$$b_1 = \frac{a_1a_2 - a_oa_3}{a_1} \quad \text{.........} \quad \mathbf{(3)}$$

$$b_2 = \frac{a_1a_4 - a_oa_5}{a_1} \quad \text{.........} \quad \mathbf{(4)}$$

$$b_3 = \frac{a_1a_6 - a_oa_7}{a_1} \quad \text{.........} \quad \mathbf{(5)}$$

$$c_1 = \frac{b_1a_3 - a_1b_2}{b_1} \quad \text{.........} \quad \mathbf{(6)}$$

$$c_2 = \frac{b_1a_5 - a_1b_3}{b_1} \quad \text{.........} \quad \mathbf{(7)}$$

$$c_3 = \frac{b_1a_7 - a_1b_4}{b_1} \quad \text{.........} \quad \mathbf{(8)}$$

$$d_1 = \frac{c_1c_2 - b_1c_2}{c_1} \quad \text{.........} \quad \mathbf{(9)}$$

$$d_2 = \frac{c_1b_3 - b_1c_3}{c_1} \quad \text{........} \quad \mathbf{(10)}$$

The process is continued until the horizontal components for s^o are obtained.

Now, Nagrath *et al*. (2009), present Routh's stability criterion as follows: "For a linear control system to be stable, it is necessary and sufficient that each term of the first column of Routh array be positive. If this condition is not met, the system is unstable and the number of sign changes of the terms of the first column corresponds to the number of roots of the characteristic equation in the right half of the s-plane (signifying instability."

Distefano III *et al*. (2010), put the stability criterion (Routh) this way: "All the roots of the characteristic equation have negative real part (signifying the system's stability), if and only if the elements of the first column of the Routh table (array) have the same sign. Otherwise, the number of roots with positive real parts (signifying instability), is equal to the number of changes of sign in the first column of the array."

Also, Ogata (2010) puts it this way: "The necessary and sufficient condition that all roots of equation (1) lie on the left half of the s-plane (signifying stability of the system), is that <u>all the coefficients of equation (1) be positive</u> and all terms in the first column of the array have positive signs."

The author of this paper researched into Routh's stability criterion and found the presentations by Nagrath *et al*. (2009) and Distefano III *et al*. (2010), to be convincingly right. However, in the presentation by Ogata (2010), the phrase ..."<u>all the coefficients of equation (1)</u>, <u>(that is the characteristic equation)</u>, <u>be positive</u>"... (which would guarantee system stability), is wrong.

The following referenced mathematical illustrations on Routh's stability criterion are presented to support this claim.

MATHEMATICAL ILLUSTRATIONS ON ROUTH'S STABILITY CRITERION

Question: Using Routh's stability criterion, determine if a control system with the characteristic equation:

$$Q(s) = s^4 + 4s^3 + 8s^2 + 16s + 32 = 0 \qquad \mathbf{(11)}$$

is stable or not (Burns, 2002).

Solution: The Routh's array is as follows:

s^4:	1	8	32
s^3:	4	16	0
s^2:	4	32	0
s^1:	-16	0	0
s^o:	32	0	0

It can be seen from the array that there is a change in sign (from positive to negative) in the first column of the array signifying that the system is unstable. Also, since the sign has changed twice (from positive to negative when transiting from s^2 to s^1, and changed again from negative to positive when transiting from s^1 to s^o), this implies that there are two real roots in the right side of the s-plane (signifying instability).

This instability condition of the control system in question would be negated by the phrase …"all the coefficients of equation (1) be positive …" in Ogata's presentation. This is because equation (1) has all its coefficients positive, yet, Routh's test has revealed that the system is unstable. Ogata's presentation is therefore incorrect.

For readers of this paper to be more convinced about this claim, the same problem is solved again below using two other stability criteria, namely the continued fraction and Hurwitz as follows:

Solution by continued fraction stability criterion:

$$Q(s) = s^4 + 4s^3 + 8s^2 + 16s + 32 = 0$$

from which,

$$Q_1(s) = s^4 + 8s^2 + 32$$

$$Q_2(s) = 4s^3 + 16s$$

$$\frac{Q_1(s)}{Q_2(s)} = \frac{s^4 + 8s^2 + 32}{4s^3 + 16s}$$

$$\begin{array}{r|l} & (\tfrac{1}{4})s \\ 4s^3 + 16s & s^4 + 8s^2 + 32 \\ & \underline{s^4 + 4s^2} \\ & \quad 4s^2 + 321 \end{array}$$

$$\frac{Q_1(s)}{Q_2(s)} = (\tfrac{1}{4})s + \frac{1}{\left(\frac{4s^3 + 16s}{4s^2 + 32}\right)}$$

working on the expression $\frac{4s^3 + 16s}{4s^2 + 32}$, we have:

$$\begin{array}{r|l} & s \\ 4s^2 + 32 & 4s^3 + 16s \\ & \underline{4s^3 + 32s} \\ & \quad -16s \end{array}$$

$$\frac{Q_1(s)}{Q_2(s)} = (\tfrac{1}{4})s + \frac{1}{s + \left(\frac{-16s}{4s^2 + 32}\right)}$$

$$\frac{Q_1(s)}{Q_2(s)} = (\tfrac{1}{4})s + \frac{1}{s + \left[\frac{-1}{4}\right] s + \frac{1}{(-\tfrac{1}{2})\, s}} \qquad \textbf{(12)}$$

Continued fraction stability criterion states that for a linear control system to be stable, all the coefficients of s in the division process should be positive (Distefano III *et al.*, 2010). From the above division process, it can be seen that the system is unstable (by the continued fraction stability criterion) hence confirming the result that was earlier obtained from Routh's stability criterion. This still negates what Ogata stated.

Solution to the illustrative problem can still be demonstrated using Hurwitz stability criterion as follows:

$$Q_1(s) = s^4 + 4s^3 + 8s^2 + 16s + 32 = 0$$

By Hurwitz's criterion, $a_4 = 1$, $a_3 = 4$, $a_2 = 8$, $a_1 = 16$, $a_o = 32$

$$\Delta_4 = \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ a_4 & a_2 & a_o & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_o \end{vmatrix} = \begin{vmatrix} 4 & 1 & 0 & 0 \\ 1 & 8 & 32 & 0 \\ 0 & 4 & 16 & 0 \\ 0 & 1 & 8 & 32 \end{vmatrix}$$

$$\Delta_4 = \left|\begin{array}{cccc:ccc} 4 & 1 & 0 & 0 & 4 & 1 & 0 \\ 1 & 8 & 32 & 0 & 1 & 8 & 32 \\ 0 & 4 & 16 & 0 & 0 & 4 & 16 \\ 0 & 1 & 8 & 32 & 0 & 1 & 8 \end{array}\right|$$

$$\Delta_4 = [(4)(8)(16)(32) + 0 + 0 + 0] - [0 + 0 + 0 + 0]$$

$$\Delta_4 = 16384 \quad \text{.......... (13)}$$

$$\Delta_3 = \left|\begin{array}{ccc:cc} 4 & 1 & 0 & 4 & 1 \\ 1 & 8 & 32 & 1 & 8 \\ 0 & 4 & 16 & 0 & 4 \end{array}\right|$$

$$\Delta_3 = [(4)(8)(16) + 0 + 0] - [0 + (4)(32)(4) + (16)(1)(1)]$$

$$\Delta_3 = 512 - [512 + 16] = -16 \quad \text{.......... (14)}$$

$$\Delta_2 = \begin{vmatrix} 4 & 1 \\ 4 & 8 \end{vmatrix} = 32 - 1 = 31 \quad \text{.......... (15)}$$

$$\Delta_1 = 4 \quad \text{.......... (16)}$$

Hurwitz's stability criterion states that for a linear control system to be stable, all the determinants of the characteristic equation (Δ's) (as shown above), must be positive (Distefano III *et al*., 2010). It can be seen here that one of the Δ's, (Δ_3), is negative. This implies (by Hurwitz stability criterion), that the control system in question is unstable. This still further negates and contradicts what Ogata stated.

DISCUSSION

The illustrative problem has been solved using three (3) different stability criteria and all the results show that the system is unstable. It can therefore be stated that Ogata (2010) was incorrect in his presentation of Routh's stability criterion. Hence, when stating Routh's stability criterion, no reference should be made to the signs of the coefficients of the characteristic equation (that is whether any is positive or negative), regarding the stability of the system, as this could be deceptive and mis-leading. The solutions to an illustrative problem just presented, where all the coefficients are positive, confirm this fact.

Besides, a modification still needs to be effected by all authors stating Routh's stability criterion. The statement that all the coefficients of the first column of Routh's array "must be positive" should be amended to read "…there must be no change in sign in the elements of the first column of the array." This is because when change in sign is emphasized, that would give fore-hand information of the number of roots of the characteristic equation in the right side of the s-plane (depending on how many times the sign changed), which would signify instability. Just saying or stating that "the signs of the first column of the array must be positive" gives no fore-hand information about the number of poles of the characteristic equation in the right side of the s-plane, (signifying instability).

Hence, Routh's stability criterion should be re-cast as follows:

"*Routh's stability criterion for linear control systems states that, for a linear control system to be stable, there should be no change in sign in the elements of the first column of Routh's array. The number of times the sign changes is the number of roots of the characteristic equation (obtained from the closed-loop transfer function of the control system), with positive real parts in the s-plane signifying instability."*

CONCLUSION

The conclusion that could be drawn from this is that while stating Routh's stability criterion, it should be as stated by the author of this paper above under DISCUSSION or as stated by Nagrath *et al*. (2009) or by Distefano III *et al*. (2010), all with the amendments as presented in the Discussion section. In subsequent edition, it is recommended that Ogata, mostly, re-phrases this criterion as presented by the author in order to remove the ambiguity and contradiction as discussed in this paper.

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