Stability Analysis of the Transmission Dynamics and Control of Corruption.

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ABSTRACT

A mathematical model for corruption as a disease in a population with constant recruitment rate incorporating standard incidence rate and effort rate against corruption was developed. The population is subdivided into four (4) different compartments according to their corruption status. The basic reproduction number (R_0) that can be used to control the transmission dynamics of the disease was obtained. Thus, the analysis revealed a globally asymptotically stable corruption-free equilibrium whenever $R_0 \leq 1$ and a globally asymptotically stable endemic equilibrium, if otherwise. Numerical simulations were carried out which confirms the analytical results and further revealed that corruption can only be curbed (reduced) to a bearable level but not totally eliminated.

(Keywords: corruption, basic reproduction number, equilibria, stability)

INTRODUCTION

Different scholars from social sciences, psychology, political sciences, and religious studies have attempted a working definition for corruption from their various disciplines. However, all of the working definitions are interwoven (Olagunju, 2012). In everyday use, corruption is a term which conveys an element of moral disapproval. The World Bank has defined corruption as "the abuse of public office for private gain" (World Bank, 1997). Additionally, Transparency International (TI) defined corruption as "the misuse of entrusted power for private benefit" (Pope, 2000). In a more legal form, Ogbu (2008) defined corruption as "an abuse of position or inducement of an abuse of position for an undeserved benefit, advantage or relief". Thus, corrupt practices includes: bribery, extortion, fraud, embezzlement, nepotism, cronyism, appropriation of public institutions assets and properties for private use, and influence peddling.

Corruption is a pressing global problem and no country in the world is totally free of its menacing grip. It is a worm within the body of the society and a cancer to economic, social, and political development. In an International Monetary Fund (IMF) working paper, Tanzi (1998) summarizes the causes or factors that promotes corruption as:

> *"those that affect demand (by the public) for corrupt acts and those that affect the supply (by public officials) of acts of corruption. Among the factors affecting the demand are (1) regulations and authorizations; (2) certain characteristics of the tax systems; (3) certain spending decisions; and (4) provision of goods and services at below-market prices. Among the factors affecting the supply of acts of corruption are (1) the bureaucratic tradition; (2) the level of public sector wages; (3) the penalty system; (4) institutional controls; (5) the transparency of rules, laws, and processes; and (6) the examples set by the leadership."*

Obtaining robust measures of corruption is a very difficult task due to the very illicit and secretive nature of corrupt practices. Though, it is possible to quantify people's perception of corruption within a society or country. Various perception indices have been developed by several institutions and have been used by various researchers. Two (2) widely used of such indices are the International Country Risk Guide (ICRG) and the Transparency International Corruption Perception Index (TICPI).

Numerous studies on corruption have been carried out due to increasing public interest and concern over its universal threat to humanity.

Although, some studies (Leff, 1964; Huntington, 1968; Friedrich, 1972; Lui, 1986; Beck and Marker, 1986; Lien, 1986) have pointed out the desirability of some corruption on the ground that it can be beneficial to the functioning of the economy, most of the studies including Myrdal (1968), Rose-Ackerman (1975), Gould and Amaro-Reyes (1983), Baumol (1990), Murphy *et al.*, (1991) and Klitgaard (1991) concluded that corruption undermines the economic, political as well as social development. It is both a major cause and a result of poverty around the world.

Several authors including Starkermann (1989), Blanchard *et al.*, (2005), Le Van and Maurel (2006), Becker *et al.*, (2008), Brianzoni *et al.*, (2011) and Waykar (2013) have over the last two and a half decades used mathematical models to evaluate the effects of corruption on national development. Modelling corruption as a diseases.

Hathroubi (2013) divided the total population (P_t) into three (3) compartments of susceptible (SC_t), corrupt (C_t), and honest (H_t) individuals, so that:

$$P_t = SC_t + C_t + H_t \tag{1}$$

where

$$\frac{dSC_t}{dt} = -iSC_t \cdot C_t \tag{2}$$

$$\frac{dC_t}{dt} = iSC_t \cdot C_t - gC_t \tag{3}$$

$$\frac{dH_t}{dt} = gC_t \tag{4}$$

Though, the stability analyses of both corruption-free and endemic equilibria were not carried out in Hathroubi (2013), an epidemiological corruption threshold based on the approximation of the honest population was determined. In this paper, we developed and analyzed a new mathematical model of corruption transmission dynamics to complement and extend the work of Hathroubi (2013).

MODEL FORMULATION

Dividing the total population into four (4) compartments of Susceptible, Corrupt, Jailed and Honest individuals we assumed that:

(a) the TICPI is an efficient way to measure people's perception of corruption of a country;

(b) a susceptible individual is one that have never engage in any corrupt practice that will have a (significantly) negative impact on his/her country's development but he/she is prone to be corrupted;

(c) a corrupt individual is one that have (even once) engage in a corrupt practice that have (significantly) negative impact on his/her country's development and capable of influencing a susceptible individual to become corrupt;

(d) a jailed individual is one that is corrupt, caught and imprison for a specified period of year(s) during which he cannot engage in any corrupt act that will have (significantly) negative impact on his/her country's development neither influence others to be corrupt;

(e) an honest individual is one that can never be corrupt no matter the condition he find himself;

(f) a susceptible, corrupt or jailed individual can willingly become honest.

where the model variables and parameters are defined as follows:

$S(t)$ Susceptible individuals at time, *t*

$C(t)$ Corrupt individuals at time, *t*

$J(t)$ Jailed individuals at time, *t*

$H(t)$ Honest individuals at time, *t*

$N(t)$ Total population at time, *t*

Λ Recruitment rate

μ Death removal rate

p Corruption transmission probability per contact

τ Effort rate against corruption, and thus $\beta = p(1-\tau)$ is the effective corruption contact rate

ϕ Rate at which corrupt individuals are caught and imprison

$\frac{1}{\psi}$ Average period jailed individuals spent in prison

θ Proportion of individuals that leaves S, C or J compartment to H compartment willingly; this may possibly be due to public enlightenment (moral and/or religious) and not due to the fear of the consequences of being caught. Thus, as the jailed individuals spent an average period of $\frac{1}{\psi}$ in prison, then $\psi\theta$ and $\psi(1-\theta)$ are the transition rates from J compartment to H compartment and back to C compartment respectively.

Figure 1 is a schematic representation of the model.

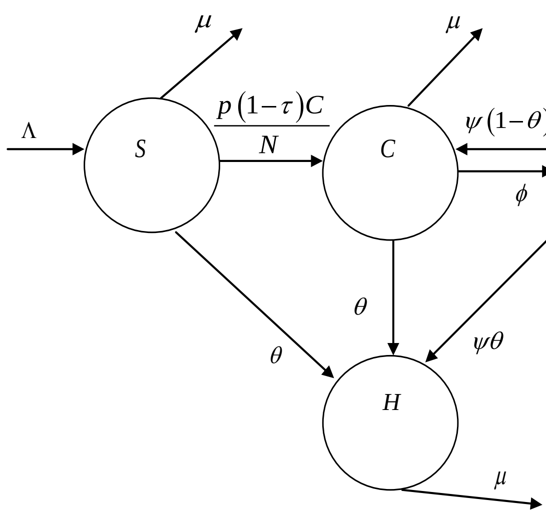

Figure 1: A Schematic Diagram of Corruption Transmission Model.

The susceptible subpopulation $S(t)$ is generated from constant recruitment of individuals at a rate Λ. They acquired infection via horizontal transfer from individuals in the corrupt class, $C(t)$ at a rate $\beta = p(1-\tau)$ and thus become corrupted. Individuals in the susceptible, corrupt or jailed class become honest due to public enlightenment (moral and/or religious) on the danger of corruption to the society at the rate $\theta\,(0<\theta<1)$. Corrupt individuals are caught and imprison at the rate ϕ. Jailed individuals stay in prison for an average period of $\frac{1}{\psi}$, after which a proportion θ become honest while $(1-\theta)$ go back to corrupt class.

Furthermore, Natural death occurs in all classes at a rate μ.

The corresponding mathematical equations of the schematic diagram can be described by a system of ordinary differential equations given in (5):

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \frac{\beta C}{N}S - K_1 S \\
\frac{dC}{dt} &= \frac{\beta C}{N}S + \psi(1-\theta)J - K_2 C \\
\frac{dJ}{dt} &= \phi C - K_3 J \\
\frac{dH}{dt} &= \theta(S + C + \psi J) - \mu H
\end{aligned} \qquad \textbf{(5)}$$

where,

$$K_1 = (\theta + \mu)$$
$$K_2 = (\phi + \theta + \mu) \quad \textbf{(6)}$$
$$K_3 = (\psi + \mu)$$

Consider the closed set:

$$\Omega = \left\{ (S, C, J, H) \in \mathbb{R}_+^4 : S + C + J + H \leq \frac{\Lambda}{\mu} \right\} \quad \textbf{(7)}$$

In order to study the dynamics of the system (5) in Ω, the positive-invariance and attractiveness of Ω with respect to the system (5) is established as follows. Now, the rate of change of the total population, obtained by adding all the equations in the system (5), is given by:

$$\frac{dN}{dt} = \Lambda - \mu N \quad \textbf{(8)}$$

It follows from (8) that whenever $N > \frac{\Lambda}{\mu}$ then, $\frac{dN}{dt} < 0$, implying $\frac{dN}{dt}$ is bounded by $\Lambda - \mu N$. Thus, a standard comparison theorem (Lakshmikanthan *et al.*, 1989) can be used to show that $N(t) \leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}\left(1 - e^{-\mu t}\right)$, in particular, $N(t) \leq \frac{\Lambda}{\mu}$ if $N(0) \leq \frac{\Lambda}{\mu}$. Thus, Ω is positively-invariant (i.e. all solutions in Ω remain in Ω for all time). Furthermore, if $N(t) > \frac{\Lambda}{\mu}$ then either the solution enters Ω in finite time or $N(t)$ approaches $\frac{\Lambda}{\mu}$ and the infected variables C and J approach zero. Thus, Ω is attracting (i.e. all solutions in $\mathbb{R}_+^4$ eventually enters Ω). Therefore, the model is well-poised epidemiologically and mathematically (Hethcote, 2000). And hence, it is sufficient to study the dynamics of the system (5) in Ω.

MODEL ANALYSIS

Existence and Local Stability of Corruption-Free Equilibrium

The Corruption-free equilibrium is the state in which the population is free of corruption, so that we have only susceptible and honest individuals. Thus, the model has a corruption-free equilibrium, obtained by setting the right-hand side of (5) to zero, given by:

$$E_0 : \left(S^*, C^*, J^*, H^*\right) = \left(\frac{\Lambda}{K_1}, 0, 0, \frac{\theta\Lambda}{\mu K_1}\right) \quad \textbf{(9)}$$

Using the next generation operator technique described by Diekmann and Heesterbeek (2000) and subsequently analysed by Van den Driessche and Watmough (2002), we obtained the basic reproduction number, R_0 of the model Equations (5) which is the spectral radius (ρ) of the next generation matrix, K. That is $R_0 = \rho K$, where $K = FV^{-1}$. The matrices of F (for the new infection terms) and V (of the transition terms) are obtained from the infected compartment (i.e., C and J) at corruption-free equilibrium and given, respectively, by:

$$F = \begin{pmatrix} \frac{\beta S^*}{N^*} & 0 \\ 0 & 0 \end{pmatrix} \quad \textbf{(10)}$$

and

$$V = \begin{pmatrix} K_2 & -\psi(1-\theta) \\ -\phi & K_3 \end{pmatrix} \quad \textbf{(11)}$$

The basic reproduction number is then given as:

$$R_0 = \frac{\beta K_3 S^*}{\left(K_2 K_3 - \phi\psi(1-\theta)\right) N^*} \quad \textbf{(12)}$$

Theorem 1: The corruption-free equilibrium, E_0 of the model is locally asymptotically stable (LAS) if $R_0 < 1$.

Proof: We used the Jacobian stability approach to prove the local stability of the corruption-free equilibrium state. Now, we observed that the variable H does not appear in the first three (3) equations of the system (5). Thus, using the relation:

$$H = N - (S + C + J) \quad \textbf{(13)}$$

allows us as explained in Hethcote (2000) and Benyah (2008) to study the first three equations of system (5). Linearization at E_0, gives the Jacobian matrix:

$$J(E_0) = \begin{pmatrix} -K_1 & -\frac{\beta S^*}{N^*} & 0 \\ 0 & -\left(K_2 - \frac{\beta S^*}{N^*}\right) & \psi(1-\theta) \\ 0 & \phi & -K_3 \end{pmatrix} \quad \textbf{(14)}$$

Considering (5) at E_0, we can deduced that

$$\begin{aligned} \Lambda &= \frac{\beta C^* S^*}{N^*} + K_1 S^* \\ K_2 &= \frac{\beta S^*}{N^*} + \frac{\psi(1-\theta)J^*}{C^*} \\ K_3 &= \frac{\phi C^*}{J^*} \\ \mu &= \frac{\theta\left(S^* + C^* + \psi J^*\right)}{H^*} \end{aligned} \quad \textbf{(15)}$$

Using elementary row transformation, equation (14) becomes:

$$J(E_0) = \begin{pmatrix} -K_1 & -\frac{\beta S^*}{N^*} & 0 \\ 0 & -\left(K_2 - \frac{\beta S^*}{N^*}\right) & \psi(1-\theta) \\ 0 & 0 & -K_3 + \frac{\phi\psi(1-\theta)}{\left(K_2 - \frac{\beta S^*}{N^*}\right)} \end{pmatrix} \quad \textbf{(16)}$$

and clearly, the eigenvalues are:

$\lambda_1 = -K_1 < 0$

$\lambda_2 = -\left(K_2 - \frac{\beta S^*}{N^*}\right) < 0$, since from equation (15) $K_2 > \frac{\beta S^*}{N^*}$; and

$$\lambda_3 = -\left(K_3 - \frac{\phi\psi(1-\theta)N^*}{\left(K_2 N^* - \beta S^*\right)}\right) \quad \textbf{(17)}$$

Now, for λ_3 to be negative, we must have $-K_3\left(K_2 N^* - \beta S^*\right) + \phi\psi(1-\theta)N^* < 0$.

Simplifying we obtained:

$$\frac{\beta K_3 S^*}{\left(K_2 K_3 - \phi\psi(1-\theta)\right)N^*} < 1$$

Thus, $\lambda_3 < 0$ if $R_0 < 1$ implying all the eigenvalues have negative real parts. And by Jacobian stability E_0 is LAS. This completes the proof.

The epidemiological implication of the theorem is that corruption can be under control (bearable) in the population (when $R_0 < 1$) if the initial sizes of the sub-populations of the model are in the basin of attraction of DFE (E_0). In order to ensure that corruption is independent of the initial sizes of the sub-populations of the model, it is necessary to show that E_0 is globally-asymptotically stable (GAS) (Garba and Gumel, 2010).

Global Stability of Corruption-Free Equilibrium

Theorem 2: The corruption-free equilibrium, E_0 of (5) is globally asymptotically stable (GAS) in Ω if $R_0 \le 1$.

Proof: One common approach in studying the global asymptotic stability of the DFE is to construct an appropriate Lyapunov function. Consider the Lyapunov function:

$$L = K_3 C + \psi(1-\theta)J \quad \textbf{(18)}$$

its derivatives along the solutions of the model equations is:

$$L' = K_3 C' + \psi(1-\theta)J' \quad \textbf{(19)}$$
$$= K_3\left(\frac{\beta C}{N}S + \psi(1-\theta)J - K_2 C\right) + \psi(1-\theta)(\phi C - K_3 J)$$
$$= C\left(K_2 K_3 - \phi\psi(1-\theta)\right)\left(\frac{K_3 \beta S}{\left(K_2 K_3 - \phi\psi(1-\theta)\right)N} - 1\right)$$

Now, since $\frac{S}{N} \le \frac{S^*}{N^*}$, we have:

$$L' \le C\left(K_2 K_3 - \phi\psi(1-\theta)\right)\left(\frac{K_3 \beta S^*}{\left(K_2 K_3 - \phi\psi(1-\theta)\right)N^*} - 1\right)$$

i.e.

$$L' \le C\left(K_2 K_3 - \phi\psi(1-\theta)\right)(R_0 - 1)$$

Since all model parameters are nonnegative and from (15) we have $K_2 K_3 > \phi\psi(1-\theta)$, it follows that when $R_0 \le 1$, $L' \le 0$; the equality $L' = 0$ holds when $R_0 = 1$ and $C = 0$. Therefore, the largest compact invariant set $\{(S, C, J, H) \in \mathbb{R}_+^4 : L' = 0\}$ is the singleton $\{E_0\}$. Hence, by the LaSalle invariance principle (LaSalle, 1976), E_0 is overall globally asymptotically stable in $\mathbb{R}_+^4$ and hence, the proof is complete.

The above theorem shows that corruption will be under control regardless of the initial profile of the subpopulation in the community if R_0 can be brought down to a level less than unity.

Existence of Corruption-Endemic Equilibrium

At the corruption-endemic equilibrium we have persistence of infection. Thus, at least one of the infected classes is greater than zero. In order to find the positive endemic equilibrium of the system (5), denoted by:

$$E^* = \left(S^{**}, C^{**}, J^{**}, H^{**}\right) \quad \textbf{(20)}$$

The equations in the system (5) are solved as explained in Gumel (2007) in terms of the associated force of infection at steady-state, given by:

$$\lambda^{**} = \frac{\beta C^{**}}{N^{**}} \quad \textbf{(21)}$$

Solving the equations of the model (5) at steady-state gives:

$$\begin{aligned} S^{**} &= \frac{\Lambda}{K_1 + \lambda^{**}}, \\ C^{**} &= \frac{\lambda^{**}\Lambda K_3}{\left(K_1 + \lambda^{**}\right)\left(K_2 K_3 - \phi\psi(1-\theta)\right)} \\ J^{**} &= \frac{\lambda^{**}\Lambda\phi}{\left(K_1 + \lambda^{**}\right)\left(K_2 K_3 - \phi\psi(1-\theta)\right)} \\ H^{**} &= \frac{\Lambda\theta\left(\left(K_2 K_3 - \phi\psi(1-\theta)\right) + \left(K_3 + \phi\psi K_3\right)\lambda^{**}\right)}{\left(K_1 + \lambda^{**}\right)\left(K_2 K_3 - \phi\psi(1-\theta)\right)\mu} \end{aligned} \quad \textbf{(22)}$$

Using the second equation of system (22) in (21) and simplifying gives:

$$\lambda^{**} = K_1\left(\frac{K_3 \beta \Lambda}{K_1\left(K_2 K_3 - \phi\psi(1-\theta)\right)N^*} - 1\right)$$

i.e.

$$\lambda^{**} = K_1(R_0 - 1) \quad \textbf{(23)}$$

Since all model parameters are assumed nonnegative with $\mu > 0$, it follows that $\lambda^{**} > 0$, whenever $R_0 > 1$. The components of E^* are then determined by substituting (23) into (22), given by:

$$S^{**}=\frac{\Lambda}{K_1R_0}$$
$$C^{**}=\frac{\Lambda K_3\left(R_0-1\right)}{\left(K_2K_3-\phi\psi\left(1-\theta\right)\right)R_0} \quad \textbf{(24)}$$
$$J^{**}=\frac{\Lambda\phi\left(R_0-1\right)}{\left(K_2K_3-\phi\psi\left(1-\theta\right)\right)R_0}$$
$$H^{**}=\frac{\Lambda\theta\left(\left(K_2K_3-\phi\psi\left(1-\theta\right)\right)+\left(K_3+\phi\psi K_3\right)\left(R_0-1\right)\right)}{\mu\left(K_2K_3-\phi\psi\left(1-\theta\right)\right)R_0}$$

Noting that $R_0<1$ implies that the force of infection at steady state $\left(\lambda^{**}\right)$ is negative (which is biologically meaningless). And hence, the model has no positive equilibria in this case. Thus, we established the following result.

Lemma 1: The system (5) has a unique endemic (positive) equilibrium whenever $R_0>1$, and no positive equilibrium otherwise.

Local Stability of Corruption-Endemic Equilibrium

Similarly, as in local stability of corruption-free equilibrium, we used the Jacobian stability approach to prove the stability of the corruption-endemic equilibrium state. Noting the relation $H=N-\left(S+C+J\right)$, the Jacobian matrix of system (5) at E^* is given by:

$$J\left(E^*\right)=\begin{pmatrix} -\left(\frac{\beta C^{**}}{N^{**}}+K_1\right) & -\frac{\beta S^{**}}{N^{**}} & 0 \\ \frac{\beta C^{**}}{N^{**}} & -\left(K_2-\frac{\beta S^{**}}{N^{**}}\right) & \psi\left(1-\theta\right) \\ 0 & \phi & -K_1 \end{pmatrix} \quad \textbf{(25)}$$

From (5) at E^*, we have:

$$\Lambda=\frac{\beta C^{**}S^{**}}{N^{**}}+K_1S^{**}$$
$$K_2=\frac{\beta S^{**}}{N^{**}}+\frac{\psi\left(1-\theta\right)J^{**}}{C^{**}}$$
$$K_3=\frac{\phi C^{**}}{J^{**}} \quad \textbf{(26)}$$
$$\mu=\frac{\theta\left(S^{**}+C^{**}+\psi J^{**}\right)}{H^{**}}$$

Using elementary row operation, (25) becomes:

$$J\left(E^*\right)=\begin{pmatrix} -\left(\frac{\beta C^{**}+K_1N^{**}}{N^{**}}\right) & -\frac{\beta S^{**}}{N^{**}} & 0 \\ 0 & -M & \psi\left(1-\theta\right) \\ 0 & 0 & -K_3+\frac{\phi\psi\left(1-\theta\right)}{M} \end{pmatrix} \quad \textbf{(27)}$$

where

$$M=\left(\left(K_2-\frac{\beta S^{**}}{N^{**}}\right)+\frac{\beta^2S^{**}C^{**}}{\left(\beta C^{**}+K_1N^{**}\right)N^{**}}\right) \quad \textbf{(28)}$$

Thus, clearly the eigenvalues are:

$$\lambda_1=-\left(\frac{\beta C^{**}+K_1N^{**}}{N^{**}}\right)<0$$
$$\lambda_2=-\left(\left(K_2-\frac{\beta S^{**}}{N^{**}}\right)+\frac{\beta^2S^{**}C^{**}}{\left(\beta C^{**}+K_1N^{**}\right)N^{**}}\right)<0$$

since from (26) $K_2>\frac{\beta S^{**}}{N^{**}}$; .and

$$\lambda_3=-\left(K_3-\frac{\phi\psi\left(1-\theta\right)}{M}\right) \quad \textbf{(29)}$$

Now, for λ_3 to be negative, we must have

$$\frac{-K_3M+\phi\psi\left(1-\theta\right)}{M}<0$$

i.e., $K_3\left(\left(K_2-\frac{\beta S^{**}}{N^{**}}\right)+\frac{\beta^2 S^{**}C^{**}}{\left(\beta C^{**}+K_1N^{**}\right)N^{**}}\right)>\phi\psi\left(1-\theta\right)$

Simplifying, we obtained: $\beta C^{**}>0$

Substituting (24) and simplifying, we obtained: $\left(R_0-1\right)>0$

Thus, $\lambda_3<0$ if $R_0>1$ implying all the eigenvalues have negative real parts. We thus, established the following result.

Theorem 3: The positive endemic equilibrium state of the system (5) is locally asymptotically stable (LAS) when $R_0>1$.

The epidemiological implication of Theorem 3 is that corruption will continue to persist in the population when $R_0>1$.

Global Stability of Corruption-Endemic Equilibrium

Theorem 4: If $R_0>1$, then the corruption-endemic equilibrium, E^* of (5) is globally asymptotically stable (GAS) in Ω when:

$$\frac{S}{S^{**}}=\frac{C}{C^{**}} \qquad \psi=1 \tag{30}$$

Proof: Consider the Lyapunov function:

$$\begin{aligned}U=&\left(S-S^{**}-S^{**}\ln\frac{S}{S^{**}}\right)+\left(C-C^{**}-C^{**}\ln\frac{C}{C^{**}}\right)+\frac{\psi\left(1-\theta\right)J^{**}}{\phi C^{**}}\left(J-J^{**}-J^{**}\ln\frac{J}{J^{**}}\right)\\&+\frac{H^{**}}{\theta}+\left(H-H^{**}-H^{**}\ln\frac{H}{H^{**}}\right)+\frac{1}{2\mu}\left[\left(S-S^{**}\right)+\left(C-C^{**}\right)+\left(J-J^{**}\right)+\left(H-H^{**}\right)\right]^2\end{aligned} \tag{31}$$

its derivatives along the solutions of the model equations is:

$$U'=\left(1-\frac{S^{**}}{S}\right)\dot{S}+\left(1-\frac{C^{**}}{C}\right)C'+\frac{\psi\left(1-\theta\right)J^{**}}{\phi C^{**}}\left(1-\frac{J^{**}}{J}\right)J'+\frac{H^{**}}{\theta}\left(1-\frac{H^{**}}{H}\right)H'+\frac{1}{\mu}\left[\begin{array}{l}\left(S-S^{**}\right)+\left(C-C^{**}\right)\\+\left(J-J^{**}\right)+\left(H-H^{**}\right)\end{array}\right]\left(S'+C'+J'+H'\right)$$

$$\begin{aligned}U'=&\left(1-\frac{S^{**}}{S}\right)\left(\Lambda-\frac{\beta CS}{N}-K_1S\right)+\left(1-\frac{C^{**}}{C}\right)\left(\frac{\beta CS}{N}+\psi\left(1-\theta\right)J-K_2C\right)+\frac{\psi\left(1-\theta\right)J^{**}}{\phi C^{**}}\left(1-\frac{J^{**}}{J}\right)\left(\phi C-K_3J\right)\\&+\frac{H^{**}}{\theta}\left(1-\frac{H^{**}}{H}\right)\left(\theta S+\theta C+\psi\theta J-\mu H\right)+\frac{1}{\mu}\left[\begin{array}{l}\left(S-S^{**}\right)+\left(C-C^{**}\right)\\+\left(J-J^{**}\right)+\left(H-H^{**}\right)\end{array}\right]\left(\Lambda-\mu\left(S+C+J+H\right)\right)\end{aligned} \tag{32}$$

Considering (8) at E^*, we can deduced that,

$$\Lambda = \mu\left(S^{**} + C^{**} + J^{**} + H^{**}\right) \quad \textbf{(33)}$$

Then, from (26) and (33), (32) becomes:

$$\begin{aligned} U' = &\left(1-\frac{S^{**}}{S}\right)\left(\frac{\beta C^{**}S^{**}}{N^{**}} + K_1 S^{**} - \frac{\beta CS}{N} - K_1 S\right) + \left(1-\frac{C^{**}}{C}\right)\left(\frac{\beta CS}{N} + \psi(1-\theta)J - \frac{\beta CS^{**}}{N^{**}} - \frac{\psi(1-\theta)CJ^{**}}{C^{**}}\right) \\ &+ \frac{\psi(1-\theta)J^{**}}{\phi C^{**}}\left(1-\frac{J^{**}}{J}\right)\left(\phi C - \frac{\phi C^{**}J}{J^{**}}\right) + \frac{H^{**}}{\theta}\left(1-\frac{H^{**}}{H}\right)\left(\theta(S+C+\psi J) - \theta\left(\frac{S^{**}H}{H^{**}} - \frac{C^{**}H}{H^{**}} - \frac{\psi J^{**}H}{H^{**}}\right)\right) \\ &+ \frac{1}{\mu}\left[\left(S-S^{**}\right) + \left(C-C^{**}\right) + \left(J-J^{**}\right) + \left(H-H^{**}\right)\right]\left(\mu\left(S^{**}+C^{**}+J^{**}+H^{**}\right) - \mu(S+C+J+H)\right) \end{aligned}$$

Let $\alpha = \dfrac{\beta}{N}, \alpha^* = \dfrac{\beta}{N^{**}}$, then, we have:

$$\begin{aligned} U' = &\frac{-K_1\left(S-S^{**}\right)^2}{S} + \alpha^* C^{**} S^{**}\left[\left(1-\frac{\alpha CS}{\alpha^* C^{**} S^{**}}\right)\left(1-\frac{S^{**}}{S}\right) + \left(\frac{\alpha CS}{\alpha^* C^{**} S^{**}} - \frac{C}{C^{**}}\right)\left(1-\frac{C^{**}}{C}\right)\right] \\ &+ \psi(1-\theta)J^{**}\left\{\left(1-\frac{C^{**}}{C}\right)\left(\frac{J}{J^{**}} - \frac{C}{C^{**}}\right) + \left(1-\frac{J^{**}}{J}\right)\left(\frac{C}{C^{**}} - \frac{J}{J^{**}}\right)\right\} + \left(\frac{H-H^*}{H}\right)\left\{\begin{array}{l}\left(SH^* - S^*H\right) \\ +\left(CH^* - C^{**}H\right) + \psi\left(JH^* - J^{**}H\right)\end{array}\right\} \\ &-\left(\left(S-S^{**}\right) + \left(C-C^{**}\right) + \left(J-J^{**}\right) + \left(H-H^*\right)\right)\left(\left(S-S^{**}\right) + \left(C-C^{**}\right) + \left(J-J^{**}\right) + \left(H-H^*\right)\right) \end{aligned}$$

Let $(x, y, z) = \left(\dfrac{S}{S^{**}}, \dfrac{C}{C^{**}}, \dfrac{J}{J^{**}}\right)$ and $g(N) = \dfrac{\alpha}{\alpha^*}$, then, we get

$$\begin{aligned} U' = -&\left\{\begin{array}{l} \dfrac{K_1 C^{**}\left(S-S^{**}\right)^2}{S} + \left(H-H^{**}\right)^2 \\ +\left[\left(S-S^{**}\right) + \left(C-C^{**}\right) + \left(J-J^{**}\right)\right]\left[\left(\left(S-S^{**}\right) + \left(C-C^{**}\right) + \left(J-J^{**}\right) + \left(H-H^{**}\right)\right)\right] \end{array}\right\} \\ &+ \alpha^* C^{**} S^{**} f_1 + \psi(1-\theta)J^{**} f_2 + f_3 \end{aligned} \quad \textbf{(34)}$$

where

$$f_1 = \left[\left(1-g(N)xy\right)\left(1-\frac{1}{x}\right) + \left(g(N)xy - y\right)\left(1-\frac{1}{y}\right)\right]; \quad f_2 = \left\{\left(1-\frac{1}{y}\right)(z-y) + \left(1-\frac{1}{z}\right)(y-z)\right\}$$

$$f_3 = \left(\frac{H-H^{**}}{H}\right)\left\{\left(SH^{**} - S^{**}H\right) + \left(CH^{**} - C^{**}H\right) + \psi\left(JH^{**}J^{**}H\right)\right\} - \left(H-H^{**}\right)\left(\left(S-S^{**}\right) + \left(C-C^{**}\right) + \left(J-J^{**}\right)\right)$$

Considering f_1, we have:

$$f_1=\left[\left(1-g(N)xy\right)\left(1-\frac{1}{x}\right)+\left(g(N)xy-y\right)\left(1-\frac{1}{y}\right)\right]=2-\frac{1}{y}-y \quad (\text{with } x=y)$$

i.e., $f_1=2-\frac{C^{**}}{C}-\frac{C}{C^{**}}$

Similarly, for f_2, we have:

$$f_2=\left\{\left(1-\frac{1}{y}\right)(z-y)+\left(1-\frac{1}{z}\right)(y-z)\right\}=2-\frac{z}{y}-\frac{y}{z}$$

i.e., $f_2=2-\frac{C^{**}J}{CJ^{**}}-\frac{CJ^{**}}{C^{**}J}$

And for f_3, with $\psi=1$, we have:

$$f_3=\left(\frac{H-H^{**}}{H}\right)\left(SH^{**}-S^{**}H\right)-\left(H-H^{**}\right)\left(S-S^{**}\right)+\left(\frac{H-H^{**}}{H}\right)\left(CH^{**}-C^{**}H\right)-\left(H-H^{**}\right)^{**}\left(C-C^{**}\right)$$
$$+\left(\frac{H-H^{**}}{H}\right)\left(JH^{**}-J^{**}H\right)-\left(H-H^{**}\right)\left(J-J^{**}\right)$$

i.e. $f_3=HH^{**}\left(S+C+J\right)\left(2-\frac{H^{**}}{H}-\frac{H}{H^{**}}\right)$

Thus, Equation (34), becomes:

$$U'=-\left\{\frac{K_1C^{**}\left(S-S^{**}\right)^2}{S}+\left(H-H^{**}\right)^2+\left[\left(S-S^{**}\right)+\left(J-J^{**}\right)+\left(C-C^{**}\right)\right]\left[\left(\left(S-S^{**}\right)+\left(C-C^{**}\right)+\left(J-J^{**}\right)+\left(H-H^{**}\right)\right)\right]\right\}$$
$$+\alpha^*C^{**}S^{**}\left(2-\frac{C^{**}}{C}-\frac{C}{C^{**}}\right)+\psi(1-\theta)J^{**}\left(2-\frac{C^{**}J}{CJ^{**}}-\frac{CJ^{**}}{C^{**}J}\right)+HH^{**}\left(S+C+J\right)\left(2-\frac{H^{**}}{H}-\frac{H}{H^{**}}\right)$$

And by the arithmetic-geometric mean inequality, the following inequalities hold:

$$\left(2-\frac{C^{**}}{C}-\frac{C}{C^{**}}\right),\left(2-\frac{C^{**}J}{CJ^{**}}-\frac{CJ^{**}}{C^{**}J}\right),\left(2-\frac{H^{**}}{H}-\frac{H}{H^{**}}\right)\le 0$$

Thus, $U'(S,C,J,H)$ is less or equal to zero with equality only if $\frac{C}{C^{**}} = \frac{J}{J^{**}} = \frac{H}{H^{**}} = 1$. Then, the largest invariant set of system (5) on the set $\{(S,C,J,H) \in \mathbb{R}^4_+ : U'(S,C,J,H) = 0\}$ is the endemic equilibrium point E^*. Thus, by LaSalle's invariance principle (LaSalle, 1976), it follows that the endemic equilibrium E^* is globally asymptotically stable when the conditions in (30) are satisfied.

Remark: As in Huo *et al.*, (2010) it is possible for the conditions in (30) to fail, in which case the global stability of E^* has not been established. However, the numerical simulation on Figure 3 seem to support the idea that E^* is still global asymptotically stable even in this case.

NUMERICAL VERIFICATION

In this section, some numerical simulations are presented to monitor the dynamics of the full system (5) for various values of the basic reproduction number in order to confirm the analytical results on the local and global stability of both the corruption-free and the endemic equilibria.

Parameter Estimation

Parameters values estimation is probably the hardest part of any mathematical modelling. This is because any small under/over estimation of a value can result in a big misleading output. Thus, the baseline values for parameters of system (5) are presented in Table 1.

Table 1: Baseline Values for Parameters of the System (5).

S/N	Parameter	Baseline Value	S/N	Parameter	Baseline Value
1	Λ	$30,000$	5	ϕ	0.0001τ
2	μ	$0.011 \le \mu \le 0.021$	6	ψ	0.143
3	p	0.036	7	θ	0.000001
4	τ	$0 < \tau < 1$	-	-	-

The recruitment rate, Λ into the susceptible sub-population is assumed to be 30,000 per year. The average lifespan of countries $\left(\frac{1}{\mu}\right)$ varies from 48.69 for Chad to 89.68 for Monaco for the year 2012 (CIA, 2013). This gives a death removal rate (μ) of 0.021 and 0.011 respectively for Chad and Monaco. Thus, $\mu = 0.013$ (the modal value) is used as hypothetical value for numerical simulations, so that $N = \frac{\Lambda}{\mu} \approx 23,307,693$.

Though, measles and Ebola have a transmission probability close to one in the absence of any preventive measure (Burckhardt, 2007), a transmission probability of corruption (p) at 0.036 is assumed in the absence of transparency and accountability. Nations effort rates against corruption (τ) varies from 0.08 for Afghanistan, North Korea and Somalia to 0.90 for Denmark, Finland and New Zealand for the year 2012 (Transparency International, 2013). Three different values of τ (0.90, 0.64 and 0.39) are used as hypothetical values for numerical simulations.

Similarly, the rate at which corrupt individuals are caught and jailed (ϕ) depends so much on the illicit and secretive nature of corruption as well as its level of prevalence in the population concerned. Thus, a direct estimation of the rate is given as $\phi = 0.0001\tau$ per year, where 0.0001 is the reduced modification value associated with the illicit and secretive nature of corruption.

While China has recently gone as far as applying the death penalty on some individuals for corruption, in most countries of the world the penalties for corruption amount to imprisonment for a time entirely at the discretion of the court (guided by the custodial sentences available for the statutory offences which ranges from 12 months to life imprisonment) or a fine or both, and/or loss of public office, in addition to repaying the amount involved or confiscating of property which is gained as a result of the criminal conduct. Though, in most countries the average period corrupt individuals spent in prison $\left(\frac{1}{\psi}\right)$ is seven (7) years, so that $\psi = 0.149$. Furthermore, as it is natural to have a very insignificant number of individuals to be honest in any human set up, a proportion $\theta = 0.000001$ of the susceptible, corrupt or jailed compartment is assumed to become honest (per year) due to public enlightenment (moral and/or religious).

One of the most important concerns about any infectious disease is its ability to invade a population. The basic reproduction number, R_0 is a measure of the potential for disease spread in a population, and is inarguably 'one of the foremost and most valuable ideas that mathematical thinking has brought to epidemic theory' (Heesterbeek and Dietz, 1996). It represents the average number of secondary cases generated by an infected individual if introduced into a susceptible population with no immunity to the disease in the absence of interventions to control the infection. If $R_0 < 1$, then on average, an infected individual produces less than one newly infected individual over the course of its infection period. In this case, the infection may die out in the long run. Conversely, if $R_0 > 1$, each infected individual produces, on average more than one new infection, the infection will be able to spread in a population. A large value of R_0 may indicate the possibility of a major epidemic.

Table 2 is a list of system (5) basic reproduction number, R_0 for sixteen selected countries arranged in ascending order. It is important to note that while τ is an increasing function of R_0, μ is a decreasing function. And thus, the higher the value of μ the lower the value of R_0 and vice versa, this can be seen in the values of R_0 for South Africa, Nigeria, India and Somalia despite the values of τ.

Table 2: Corruption-Basic Reproduction Number, R_0 for Sixteen (16) Selected Countries.

S/N	Country	τ	μ	R_0	S/N	Country	τ	μ	R_0
1	Denmark	0.9	0.013	0.28	9	Nigeria	0.27	0.018	1.46
2	Canada	0.84	0.012	0.48	10	India	0.36	0.015	1.54
3	U.S.A.	0.73	0.013	0.75	11	Saudi Arabia	0.44	0.013	1.55
4	U.K.	0.74	0.012	0.78	12	Somalia	0.08	0.020	1.66
5	South Africa	0.43	0.020	1.03	13	China	0.39	0.013	1.69
6	Israel	0.60	0.012	1.20	14	Russia	0.28	0.015	1.73
7	Georgia	0.52	0.013	1.33	15	Iran	0.28	0.014	1.85
8	Cuba	0.48	0.013	1.44	16	Uzbekistan	0.17	0.013	2.30

Numerical Simulations

Using the parameters value in Table 1, with $\mu = 0.013$ and $\tau = 0.90$, so that $R_0 = 0.28 < 1$, it is seen clearly from Figure 1 that the solution profiles of system (5) converges to the corruption-free equilibrium in all the three different initial values of $C(0)$. This confirms the analytical results of the local as well as the global asymptotic stability of the corruption-free equilibrium. It is also observed from the figure that despite high value of τ (effort rate against corruption), the total number of corrupt individuals did not drop to 0 from 150 even at 100 years. This shows clearly that unlike small pox the illicit and secretive nature of corruption can never allowed for its total eradication, but can be curb (reduce) to a tolerable level.

Similarly, using the parameters value in Table 1, with $\mu = 0.013$ and $\tau = 0.64$, so that $R_0 = 1$, it is seen from Figure 2 that the solution profiles of system (5) gradually at long run converges to the corruption-free equilibrium in all the three different initial values of $C(0)$. The figure clearly revealed that $R_0 = 1$ is the margin between corruption-free and endemic equilibrium. Thus, the analytical results of the local as well as the global asymptotic stability of the corruption-free equilibrium are confirmed.

Furthermore, with $\tau = 0.39$ and $\mu = 0.013$, using the parameters value in Table 1, so that $R_0 = 1.69 > 1$, it is seen clearly from Figure 3 that the solution profiles of system (5) converges to the endemic equilibrium in all the three different initial values of $C(0)$. This confirms the analytical results of the local as well as the global asymptotic stability of the endemic equilibrium. Additionally, it is observed that despite $\tau \neq 1$ ($\tau = 0.149$), the total number of corrupt individuals continued to rise. This supports the idea that the endemic equilibrium of system (5) is still globally asymptotically stable, even if Equation (30) is not satisfied.

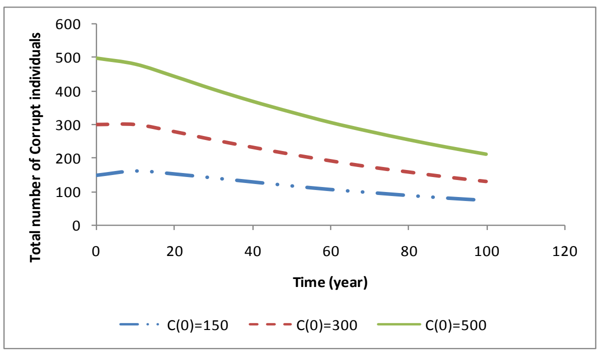

Figure 1: Total number of corrupt individuals with different initial population of $C(0) = 150, 300 \text{ and } 500$. Parameters value used are as in Table 1 with $\tau = 0.90$ and $\mu = 0.013$, so that $R_0 = 0.28$. This confirms the local as well as the global asymptotic stability of the corruption-free equilibrium.

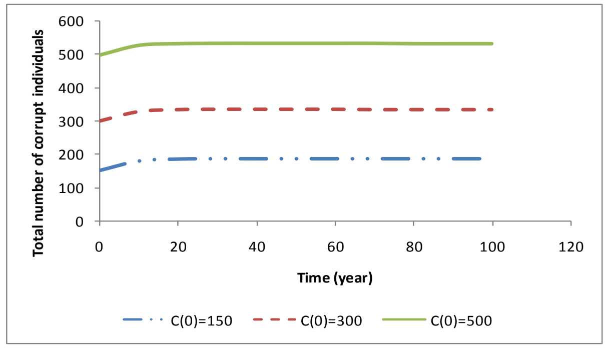

Figure 2: Total number of corrupt individuals with different initial population of $C(0) = 150, 300 \text{ and } 500$. Parameters value used are as in Table 1 with $\tau = 0.64$ and $\mu = 0.013$, so that $R_0 = 1$. This confirms the local as well as the global asymptotic stability of the corruption-free equilibrium.

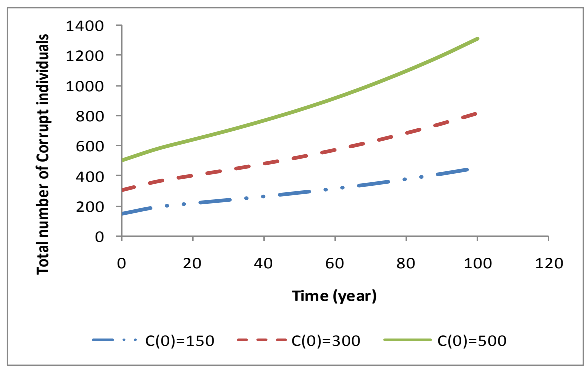

Figure 3: Total number of corrupt individuals with different initial population of $C(0) = 150, 300 \text{ and } 500$. Parameters value used are as in Table 1 with $\tau = 0.39$ and $\mu = 0.013$, so that $R_0 = 1.69$. This confirms the local as well as the global asymptotic stability of the endemic equilibrium.

CONCLUSION

A mathematical model with constant recruitment rate and standard incidence for the transmission dynamics of corruption as a disease was proposed. The basic reproduction number (R_0) was obtained and the analysis revealed that for $R_0 \leq 1$, the corruption-free equilibrium is globally asymptotically stable. Although, the illicit and secretive nature of corruption can never allow for its total eradication, but it can be curbed (reduced) to a bearable level. And for whatever reason if $R_0 > 1$ the corruption-free equilibrium point is unstable and the endemic equilibrium emerges.

Finally, there is need to apply the model to a country of interest incorporating High Anti-Corruption Feasible Reforms (HAPR), which include among other things: public awareness on the danger and consequence of corruption, transparency, accountability and very harsh effective penalties. Though, HAPR is possible only when the government is willing to curb corruption, poverty level is reduced and public sector wages increased (where and when necessary). The author has started a work titled “Curtailing Corruption in Nigeria: A Mathematical Modelling Approach”.

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