Derivation of Block Hybrid Method for the Solution of First Order Initial Value Problems in ODEs.

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ABSTRACT

This paper is concerned with the derivation and implementation of hybrid linear multistep method (LMM) for solving first order differential equations. The continuous and discrete schemes for k=4 with one off-step point at interpolation were derived, where k is the step number of the method. The continuous hybrid formulations were evaluated at various points to obtain discrete schemes, which were used in block form for parallel or sequential solution of initial value problem (IVP). For acceptability, the schemes so derived in block form were tested for consistence, zero stability and convergence. Also provided are examples of initial value problems solved with the proposed schemes in block form.

(Keywords: block method, linear multistep method, multistep collocation, continuous multistep (CM), self-starting, zero-stability)

INTRODUCTION

The hybrid schemes have been developed since the 1960's but these methods have not yet received a great deal of attention. Lie and Norsett (1989), Onumanyi., et al (1994), Yahaya and Mohammed (2010) , Yahaya (2004) and Mohammed (2010) have all converted conventional linear multistep methods including hybrid ones into continuous forms through the idea of Multistep Collocation (MC). The Continuous Multistep (CM) method, associated with conventional linear multistep methods produces piece-wise polynomial solutions over k steps for the first order differential system.

This research work aims at deriving a four-step block hybrid method for numerical integration of ordinary differential equations. It allows the block formulation and therefore is self-starting and for appropriate choice of k, overlap of solution model is eliminated.

Derivation of the Continuous and Discrete Block Hybrid Methods

Using the general multistep collocation methods see (Onumanyi, et al., 1994), (Yahaya and Mohammed, 2010), and (Mohammed, 2010) lead to the following D-matrix;

$$D=\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ 1 & x_{n+\mu} & x_{n+\mu}^2 & x_{n+\mu}^3 & x_{n+\mu}^4 & x_{n+\mu}^5 & x_{n+\mu}^6 & x_{n+\mu}^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 \end{pmatrix}$$

Using maple software package gives the column of D^{-1} which are the elements of the matrix C. the elements of C are then used to generate the value of continuous coefficient:

$$\alpha_1(x), \alpha_2(x), \alpha_3(x), \alpha_v(x), \beta_0(x), \beta_1(x), \beta_2(x), \beta_3(x) \qquad \textbf{(1)}$$

The values of the continuous coefficient (1) are substituted to give the continuous form of the four-step block hybrid methods with one off step point at interpolation.

$$
\begin{aligned}
\bar{y}(x) = \Big[& 19(2\mu^2 - 8\mu - 3)(x - x_n)^7 - (38\mu^3 + 304\mu^2 - 1872\mu - 702)(x - x_n)^6 h \\
& + 3(152\mu^3 + 60\mu^2 - 2864\mu - 1074)(x - x_n)^5 h^2 \\
& - 5(399\mu^3 - 818\mu^2 - 3616\mu - 1356)(x - x_n)^4 h^3 \\
& + (3800\mu^3 - 12122\mu^2 - 17112\mu - 6417)(x - x_n)^3 h^4 \\
& - (-738 - 3444\mu^2 - 1968\mu + 912\mu^3)(x - x_n)^2 h^5 \\
& + (-2214\mu^2 + 513\mu^3) h^7 \Big] y_n \Big/ 27h^7(19\mu - 82)\mu^2 \\
+ \Big[& -(4\mu^2 - 27\mu + 42)(x - x_n)^7 + (4\mu^3 + 24\mu^2 - 294\mu + 500)(x - x_n)^6 h \\
& - 3(17\mu^3 - 32\mu^2 - 345\mu + 718)(x - x_n)^5 h^2 \\
& + 5(792 - 208\mu^2 - 180\mu + 48\mu^3)(x - x_n)^4 h^3 \\
& - (495\mu^3 - 2676\mu^2 + 1728\mu + 2592)(x - x_n)^3 h^4 \\
& + 3(-744\mu^2 + 864\mu + 126\mu^3)(x - x_n)^2 h^5 \Big] y_{n+1} \Big/ 4h^7(19\mu - 82)(\mu - 1)^2 \\
+ \Big[& -(44\mu^2 - 359\mu + 738)(x - x_n)^7 + (44\mu^3 + 88\mu^2 - 2886\mu + 7380)(x - x_n)^6 h \\
& - 3(149\mu^3 - 708\mu^2 - 1565\mu + 8118)(x - x_n)^5 h^2 \\
& + 5(5904 - 2048\mu^2 - 1928\mu + 300\mu^3)(x - x_n)^4 h^3 \\
& - (1835\mu^3 - 1397\mu^2 + 23616\mu + 11808)(x - x_n)^3 h^4 \\
& + 3(-1968\mu^2 + 3936\mu + 246\mu^3)(x - x_n)^2 h^5 \Big] y_{n+3} \Big/ h^7(19\mu - 82)(\mu^2 + 3 - 4\mu)^2 \mu^2 \\
+ \Big[& 19(x - x_n)^7 - 234(x - x_n)^6 h + 1074(x - x_n)^5 h^2 - 2260(x - x_n)^4 h^3 \\
& + 2139(x - x_n)^3 h^4 - 738(x - x_n) h^5 \Big] y_{n+\mu} \Big/ h^7(19\mu - 82)(\mu^2 + 3 - 4\mu)^2 \mu \\
+ \Big[& -(35\mu - 152)(x - x_n)^7 - (35\mu^2 + 280\mu - 1872)(x - x_n)^6 h \\
& + 3(144\mu^2 + 38\mu - 2864)(x - x_n)^5 h^2 \\
& - 5(-18080 + 1986\mu^2 - 4408\mu)(x - x_n)^4 h^3 \\
& + (4184\mu^2 - 14117\mu - 17112)(x - x_n)^3 h^4 \\
& + (-5904\mu + 1368\mu^2)(x - x_n) h^6 \\
& - 3(1321\mu^2 - 5248\mu - 1968)(x - x_n)^2 h^5 \Big] f_n \Big/ h^7(19\mu - 82)(\mu^2 + 3 - 4\mu)^2 \mu \\
+ \Big[& (17\mu - 74)(x - x_n)^7 - (17\mu^2 + 119\mu - 838)(x - x_n)^6 h \\
& + (193\mu^2 - 65\mu - 3350)(x - x_n)^5 h^2 - (773\mu^2 - 2087\mu - 5466)(x - x_n)^4 h^3 \\
& + (1263\mu^2 - 4800\mu - 2880)(x - x_n)^3 h^4 \\
& - (-2880\mu + 666\mu^2)(x - x_n)^2 h^5 \Big] f_{n+1} \Big/ 12h^6(19\mu - 82)(\mu - 1) \\
+ \Big[& (13\mu - 58)(x - x_n)^7 - (13\mu^2 + 65\mu - 546)(x - x_n)^6 h \\
& + 3(41\mu^2 - 55\mu - 562)(x - x_n)^5 h^2 \\
& - (381\mu^2 - 1241\mu - 1966)(x - x_n)^4 h^3 \\
& + (444\mu^2 - 1792\mu - 768)(x - x_n)^3 h^4 \\
& - 3(-256\mu + 58\mu^2)(x - x_n)^2 h^5 \Big] f_{n+3} \Big/ 36h^6(19\mu - 82)(\mu - 3) \\
+ \Big[& -(x - x_n)^7 + (\mu + 8)(x - x_n)^6 h - (8\mu + 22)(x - x_n)^5 h^2 \\
& + (24 + 22\mu)(x - x_n)^4 h^3 - (24\mu + 9) \\
& + (x - x_n)^3 h^4 + 9\mu(x - x_n)^2 h^5 \Big] f_{n+4} \Big/ 24h^6(19\mu - 82)
\end{aligned}
$$

(2)

Evaluating (2) at point

$$x = x_{n+2}, x = x_{n+4}, x = x_{n+\frac{3}{2}}, \ \mu = \frac{7}{2}$$

and its derivative at point

$$x = x_{n+2}, x = x_{n+\frac{3}{2}}, x = x_{n+\frac{7}{2}}$$

yield the following six discrete hybrid method which are used as a block integrator,

$$
\begin{aligned}
& y_{n+4} + \frac{145}{1519} y_n + \frac{8}{775} y_{n+1} + \frac{56}{31} y_{n+3} - \frac{110592}{37975} y_{n+\frac{7}{2}} \\
& = \frac{h}{1085}\left[-30 f_n - 168 f_{n+1} - 840 f_{n+3} + 210 f_{n+4}\right] \\
& y_{n+2} - \frac{2519}{13671} y_n - \frac{228}{775} y_{n+1} + \frac{20}{279} y_{n+3} - \frac{22528}{37975} y_{n+\frac{7}{2}} \\
& = \frac{h}{6510}\left[325 f_n + 2688 f_{n+1} - 392 f_{n+3} - 105 f_{n+4}\right] \\
& y_{n+\frac{3}{2}} - \frac{1615}{12152} y_n - \frac{567}{775} y_{n+1} + \frac{23}{248} y_{n+3} - \frac{8667}{37975} y_{n+\frac{7}{2}} \\
& = \frac{h}{138880}\left[4785 f_n + 59346 f_{n+1} - 28770 f_{n+3} - 945 f_{n+4}\right] \\
& y_{n+1} + \frac{10400}{60858} y_n - \frac{43610}{60858} y_{n+3} - \frac{27648}{60858} y_{n+\frac{7}{2}} \\
& = \frac{h}{60858}\left[2541 f_n + 37926 f_{n+1} - 82026 f_{n+2} + 43806 f_{n+3} + 441 f_{n+4}\right] \\
& y_{n+3} - \frac{606800}{509600} y_n + \frac{1714608}{509600} y_{n+1} - \frac{1617408}{509600} y_{n+\frac{7}{2}} \\
& = \frac{h}{509600}\left[165025 f_n + 939330 f_{n+1} - 1944320 f_{n+3/2} - 1531250 f_{n+3} - 46305 f_{n+4}\right] \\
& y_{n+\frac{7}{2}} - \frac{758000}{31090176} y_n - \frac{148176}{31090176} y_{n+1} - \frac{30184000}{31090176} y_{n+3} \\
& = \frac{h}{31090176}\left[217875 f_n + 1265670 f_{n+1} + 9518250 f_{n+3} \right. \\
& \qquad \left. + 7499520 f_{n+\frac{7}{2}} - 385875 f_{n+4}\right] \qquad (3)
\end{aligned}
$$

Equation (3) constitute the member of a zero-stable block integrators of order (7,7,7,7,7,7)with $\left[-\frac{81587}{260400}, -\frac{81587}{260400}, -\frac{4966677}{44441600}, \frac{796943889}{64}, \frac{721161}{32}, \frac{52502877}{64}\right]^T$ the application of the block integrators with n=0 give the accurate values of y_1, y_2, y_3 along with $y_4$as shown in Tables 1-4. To start the IVP integration on the sub-interval $[x_0, x_4]$. We compute(3), when n=0 i.e. the 1-block 4 point method as given in Equation (4) produces its unknown simultaneously without recourse to any starting method (predictor) to generate y_1, y_2, y_3 before computing y_4.

Convergence Analysis of Block Hybrid Methods

Recall, that, it is a desirable property for a numerical integrator to produce solution that behave similar to the theoretical solution to a problem at all times. Thus several definitions, which call for the method to possess some “adequate” region of absolute stability, can be found in several literatures. See (Lambert, 1973), (Fatunla,1992), and (Fatunla, 1994), etc. following (Funtula, 1992), the four integrator proposed in this report in Equations (3) is put in the matrix-equation form and for easy analysis the result was normalized to obtain;

$$\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\\y_{n+3}\\y_{n+\frac{7}{2}}\\y_{n+4}\end{pmatrix}=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n-5}\\y_{n-4}\\y_{n-3}\\y_{n-2}\\y_{n-1}\\y_{n}\end{pmatrix}$$

$$+h\begin{pmatrix}\frac{4957}{2520}&-\frac{2318}{945}&\frac{1963}{1260}&-\frac{4807}{7560}&\frac{806}{2205}&-\frac{347}{5040}\\ \frac{9693}{4480}&-\frac{1159}{560}&\frac{6561}{4480}&-\frac{2729}{4480}&\frac{1377}{3920}&-\frac{297}{4480}\\ \frac{674}{315}&-\frac{1696}{945}&\frac{544}{315}&\frac{442618}{2125305}&\frac{160}{441}&-\frac{43}{630}\\ \frac{621}{280}&-\frac{74}{35}&\frac{351}{140}&-\frac{13}{280}&\frac{54}{245}&-\frac{27}{560}\\ \frac{12691}{5760}&-\frac{4459}{2160}&\frac{14063}{5760}&\frac{2507069}{9715680}&\frac{329}{720}&-\frac{343}{5760}\\ \frac{704}{315}&-\frac{2048}{945}&\frac{808}{315}&\frac{64}{945}&\frac{2048}{2205}&\frac{34}{315}\end{pmatrix}\begin{pmatrix}f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\\f_{n+3}\\f_{n+\frac{7}{2}}\\f_{n+4}\end{pmatrix}$$

$$+h\begin{pmatrix}0&0&0&0&0&\frac{28267}{105840}\\0&0&0&0&0&\frac{2083}{7840}\\0&0&0&0&0&\frac{3523}{13230}\\0&0&0&0&0&\frac{1033}{3920}\\0&0&0&0&0&\frac{1141}{4320}\\0&0&0&0&0&\frac{1738}{6615}\end{pmatrix}\begin{pmatrix}f_{n-5}\\f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_{n}\end{pmatrix} \quad \textbf{(4)}$$

The first characteristic polynomial of the block hybrid method is given by:

$$\rho(R)=\det\left(RA^{0}-A^{1}\right)$$

Substituting the value of A^0 and A^1 into the function above gives:

$$\rho(R)=\det\left[R\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{pmatrix}\right]$$

$$=\det\begin{pmatrix}R&0&0&0&0&-1\\0&R&0&0&0&-1\\0&0&R&0&0&-1\\0&0&0&R&0&-1\\0&0&0&0&R&1\\0&0&0&0&0&R-1\end{pmatrix}$$

$$=\left[R^{5}(R-1)\right]$$

Therefore, $R=0,\ R=1.$ The hybrid method is zero stable and consistence since the order of the method $p = 7 > 1$. And by (Henrici, 1962), the block hybrid method is convergent.

Numerical Experiment

In this paper we use newly constructed block hybrid methods and four step block hybrid Adams-mouton methods proposed by (Yahaya and Sokoto, 2010) to solve stiff and non-stiff initial value problems (IVP), in order to test for efficiency of the schemes derived.

Example 1

Consider the initial value problem

$$\begin{aligned}&y'=-y, && y(0)=1\\ &0\le x\le 1 && h=0.1\\ &exact\ \ solution: && y(x)=e^{-x}\end{aligned} \quad \textbf{(5)}$$

Example 2

Consider the initial value problem

$$\begin{aligned}&y'=-9y, && y(0)=1\\ &0\le x\le 1 && h=0.1\\ &exact\ \ solution: && y(x)=e^{1-9x}\end{aligned} \quad \textbf{(6)}$$

Firstly we transform the schemes by substitution, to get a recurrence relation. Substituting n=0, 4... and solving simultaneously using maple software package we obtain the required results displayed in tables below.

Table 1: Example 1.

X	Exact Solution	Block Hybrid Method for k=4, off-grid at $x_{n+7/2}$	Block Hybrid Adams-Moulton Method for k=4
0.1	0.9048374180	0.9048374180	0.9048374173
0.2	0.8187307531	0.8187307531	0.8187307526
0.3	0.7408182207	0.7408182205	0.7408182202
0.4	0.6703200460	0.6703200461	0.6703200456
0.5	0.6065306597	0.6065306603	0.6065306588
0.6	0.5488116361	0.5488116368	0.5488116354
0.7	0.4965853038	0.4965853042	0.4965853031
0.8	0.4493289641	0.4493289649	0.4493289635
0.9	0.4065696597	0.4065696606	0.4065696588
1.0	0.3678794412	0.3678794420	0.3678794404

Table 2: Comparison of Absolute Error for Example 1 (non-stiff).

X	Exact Solution	Block Hybrid Method for k=4, off-grid at $x_{n+7/2}$	Block Hybrid Adams-Moulton Method for k=4
0.1	0.9048374180	0	7.36E-10
0.2	0.8187307531	0	4.78E-10
0.3	0.7408182207	2.000E-10	4.82E-10
0.4	0.6703200460	1.000E-10	4.36E-10
0.5	0.6065306597	6.000E-10	9.13E-10
0.6	0.5488116361	7.000E-10	6.94E-10
0.7	0.4965853038	4.000E-10	6.91E-10
0.8	0.4493289641	8.000E-10	6.17E-10
0.9	0.4065696597	9.000E-10	9.41E-10
1.0	0.3678794412	8.000E-10	7.71-E10

Table 3: Example 2.

X	Exact Solution	Block Hybrid Method for k=4, off-grid at $x_{n+7/2}$	Block Hybrid Adams-Moulton Method for k=4
0.1	1.10517E+00	1.10501E+00	1.10399442
0.2	4.49329E-01	4.49262E-01	0.449177460
0.3	1.82684E-01	1.82678E-01	0.182461349
0.4	7.42736E-02	7.42874E-02	0.074136190
0.5	3.01974E-02	3.01985E-02	0.030109439
0.6	1.22773E-02	1.22778E-02	0.012250498
0.7	4.99159E-03	4.99238E-03	0.004976301
0.8	2.02943E-03	2.03019E-03	0.002021930
0.9	8.25105E-04	8.25289E-04	0.000821180
1.0	0.3678794412	3.35538E-04	0.000334110

Table 4: Comparison of Absolute Error for Example 2 (stiff).

X	**Exact Solution**	**Block Hybrid Method for k=4, off-grid at $x_{n+7/2}$**	**Block Hybrid Adams-Moulton Method for k=4**
0.1	1.10517E+00	1.64936E-04	1.18E-03
0.2	4.49329E-01	6.70433E-05	1.62E-04
0.3	1.82684E-01	5.32080E-06	2.22E-04
0.4	7.42736E-02	1.38386E-05	1.37E-04
0.5	3.01974E-02	1.11920E-06	8.79E-05
0.6	1.22773E-02	4.55150E-07	5.23E-04
0.7	4.99159E-03	7.84514E-07	1.53E-05
0.8	2.02943E-03	7.55940E-07	1.64E-06
0.9	8.25105E-04	1.84229E-07	3.92E-06
1.0	0.3678794412	1.99127E-06	1.35E-06

Consider the absolute errors of the two methods above. A close observation of Table 2 shows the discrete scheme of the newly constructed block hybrid methods performs far better than the standard Adams-Moulton methods when applies to non-stiff equations. While a close observation of Tables 1 and 2 also show the discrete scheme of the newly constructed block methods performs little better than the standard Adams-Moulton methods when applies to stiff equations.

CONCLUSIONS

A collocation technique which yields a method with continuous coefficients has been presented for the approximate solution of first order ODEs with initial conditions. Two test examples have been solved to demonstrate the efficiency of the proposed methods and the results compare favourably with the exact solution and four step block hybrid Adams-Moulton methods.

Interestingly, all the discrete schemes used in the Block formulation were from a single continuous formulation (CF).

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