A Non-Classical Finite Difference Method for Solving Two Point Boundary Value Problems.

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ABSTRACT

A new kind of finite difference method is presented for solving two point boundary value problems. The method is based on rational function approximation technique and its development and analysis is based on Taylor series and binomial expansions. Local truncation error of the method is discussed. The method is tested on model problems and the numerical results are presented. The numerical results show that the method is effective and accurate.

(Keywords: non-classical difference method, boundary value problem, rational approximation)

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INTRODUCTION

In this paper we consider numerical solution of the boundary value problem:

$$y'' = f(x, y)\ ,\quad x \in [a, b] \subset \mathbb{R}, y(x), f(x, y) \in \mathbb{R} \qquad (1)$$

subject to the boundary conditions:

$y(a) = y_0$, and $y(b) = y_n$, y_0, y_n are finite constants.

The existence and uniqueness of the solution for the problem (1) is assumed. The specific assumption on $f(x, y)$,to ensure the existence and uniqueness will not be considered [1,2,3].

Much research has been done on the subject of numerical solution of two point boundary value problems. In the literature for solving two point boundary value problems, usually the adopted methods are finite differences, shooting, adomian decomposition, quasi linearization, etc. The difference methods are applicable to wider classes of problems and usually the most convenient for computer solution. However all classical finite difference methods are based on the polynomial functions, which are normally smooth and with sufficient continuous derivatives. Thus, these methods are exact for some degree of polynomials.

In this paper a new finite difference method is proposed for the solution of two point boundary value problems. The method is based on rational function approximation [4,5,6]. We examine the method in order to appreciate the concept in them and underlying merits or demerits.

In this paper we out line our non-classical finite difference method, discuss the local truncation error, illustrate the proposed method by some model problems, and offer conclusions.

DERIVATION OF THE METHOD

An idea of rational approximation method for solution of two point boundary value problems and development can be found in the literature [7,8,9].

The first step in obtaining the propose rational method is to partition the interval [a , b] in which the solution of the problem (1) is desired, into N ,a finite number of subintervals [a,x_1],[x_1,x_2], [x_2,x_3],……. [x_{N-1} , b] by the points,

$$x_j = x_0 + jh\ ,\quad j = 0,1,2, \ldots\ldots\ldots N \qquad (2)$$

where the terms on right hand side in (2) are defined as:

$$\textit{the step length}\ \ h = \frac{(b-a)}{N}\ ,\quad x_0 = a \ \ \textit{and}\ x_N = b\ .$$

Suppose we have to determine a number y_i, which is an approximation to the value of the theoretical solution $y(x)$ of the problem (1) at the nodal point $x_i, i = 1,2,\dots, N-1$. Assuming local assumption that no previous truncation errors have been made [10] i.e. $y_n = y(x_n)$, we are interested in obtaining an approximate value y_{n+1}, for the value of the theoretical solution $y(x_{n+1})$. For that purpose similar to given in [5,10,12] ,we propose an approximation to the theoretical solution $y(x_n + h)$ of the problem (1) as:

$$y_{i+1} - 2y_i + y_{i-1} = \frac{h^2 y_i''}{G(h)+a_0 y_{i+1}+a_1 y_i+a_2 y_{i-1}} \qquad (3)$$

where $G(h)$,an unknown sufficiently differentiable function of step length h and coefficients a_0, a_1, a_2 are unknown constants. We put some restrictions, in determination of these constants. Also we assumed that :

$$G(h) + a_0 y_{i+1} + a_1 y_i + a_2 y_{i-1}$$

will not vanish for any i=1,2,.. …, N .

Let us define a function $F_i(a, h, y)$ and associate (3) with it as,

$$F_i(a,h,y) \cong \big(y(x_i+h) - 2y(x_i) + y(x_i - h)\big) \times \big(G(h) + a_0 y(x_i+h) + a_1 y(x_i) + a_2 y(x_i - h)\big) - h^2 y''(x_i) = 0 \qquad (4)$$

We assumed that the function $G(h)$ can be expanded in Taylor series. So we write Taylor series expansion for the function $G(h)$ about $h = 0$,

$$G(h) = G(0) + h\,G'(0) + \frac{h^2}{2!}G''(0) + \frac{h^3}{3!}G'''(0) + o(h^4) \qquad (5)$$

Expand $F_i(a, h, y)$ in Taylor series about point x_i, we have:

$$\Big(h^2 y_i'' + \frac{h^4}{12} y_i^{(4)} + o(h^6)\Big) \times \Big(G(h) + (a_0+a_1+a_2)y_i + h(a_0 - a_2)y_i' + \frac{h^2}{2!}(a_0+a_2)y_i''$$

$$+ \frac{h^3}{3!}(a_0 - a_2)y_i''' + \frac{h^4}{4!}(a_0+a_2)y_i^{(4)} + o(h^5)\Big) - h^2 y_i'' = 0 \qquad (6)$$

Using (5) in (6) and rewrite as,

$$h^2\big(G(0) + (a_0+a_1+a_2)y_i\big)y_i'' + h^3\big(G'(0) + (a_0 - a_2)y_i'\big)y_i'' + \frac{h^4}{12}\Big(\big(G(0) + (a_0+a_1+a_2)y_i\big)y_i^{(4)} + 6\big(G''(0) + (a_0+a_2)y_i''\big)y_i''\Big) + \frac{h^5}{12}\Big(\big(G'(0) + (a_0 - a_2)y_i'\big)y_i^{(4)} + 2\big(G'''(0) + (a_0 - a_2)y_i'''\big)y_i''\Big) + o(h^6) = h^2 y_i'' \qquad (7)$$

Compare the coefficients of same power of h, on both side of (7), we will get:

$$G(0) + (a_0+a_1+a_2)y_i = 1 \qquad (8.1)$$

$$G'(0) + (a_0 - a_2)y_i' = 0 \qquad (8.2)$$

$$\big(G(0) + (a_0+a_1+a_2)y_i\big)y_i^{(4)} + 6\big(G''(0) + (a_0+a_2)y_i''\big)y_i'' = 0 \qquad (8.3)$$

$$\big(G'(0) + (a_0 - a_2)y_i'\big)y_i^{(4)} + 2\big(G'''(0) + (a_0 - a_2)y_i'''\big)y_i'' = 0 \qquad (8.4)$$

Solving system of linear equation in which, $G(0), G'(0), G''(0), G'''(0), a_0$,a_1, and a_2

considered as unknowns. If we assume that $a_0 = a_2$, so from (8.2) and (8.4) ,we conclude $G'(0) = G'''(0) = 0$. Thus solution of equations (8) ,we obtained:

$$G(0) = 1 - (2a_0+a_1)y_i \qquad (9.1)$$

$$G''(0) = -2a_0 y_i'' - \frac{y_i^{(4)}}{6y_i''} \qquad (9.2)$$

Substituting (9) in (5), neglecting the term $o(h^4)$,we can write $G(h)$ as,

$$G(h) = 1 - (2a_0+a_1)y_i - \frac{h^2}{2!}\left(2a_0 y_i'' + \frac{y_i^{(4)}}{6y_i''}\right) \qquad (10)$$

Substitute (10) in (3),neglecting the term $o(h^4)$ in denominator, we obtained:

$$y_{i+1} - 2y_i + y_{i-1} = \frac{12 h^2 (y_i'')^2}{12 y_i'' - h^2 y_i^{(4)}} \qquad (11)$$

If we replace $y_i^{(4)}$ by second order difference formulae

$\frac{y''_{i+1} - 2y''_i + y''_{i-1}}{h^2}$ in (11) ,we have:

$$y_{i+1} - 2y_i + y_{i-1} = \frac{12.h^2(y''_i)^2}{14.y''_i - y''_{i+1} - y''_{i-1}} \qquad (12)$$

where y_i is an approximation to the value of the theoretical solution $y(x)$ of the problem (1) at the nodal point $x = x_i$. A similar notation we can define for y''_i i.e.$y''_i = f(x_i, y_i) = f_i$ .Introduce the notation so defined in (12) ,we get our rational approximation method for solving numerically problem (1) as,

$$y_{i+1} - 2y_i + y_{i-1} = \frac{12.h^2(f_i)^2}{14.f_i - f_{i+1} - f_{i-1}} \qquad (13)$$

The resulting system of Equations (13) are linear /nonlinear depends on $f(x, y)$. If source function f has dependent variable as an argument then (13) will be non-linear otherwise it will be linear. Thus system of linear equations may be solved by direct method and system of nonlinear equations solved by Newton-Raphson method. Computational results confirmed the performance, effectiveness and accuracy of the proposed method. Otherwise let us observe the other advantage with rational method (11) or (13). Numerov's method is the highest order method which is at same time a three point method as our method (13). Since Runge-Kutta method is lower order method, so it is better to discuss method (13) with highest order method. Let we write (13) as:

$$y_{i+1} - 2y_i + y_{i-1} = \frac{12.h^2(f_i)^2}{14.f_i}\left(1 - \frac{f_{i+1}}{14.f_i} - \frac{f_{i-1}}{14.f_i}\right)^{-1} \qquad (14)$$

expand (14) in binomial expansion , we have:

$$y_{i+1} - 2y_i + y_{i-1} = \frac{12.h^2 f_i}{14}\left(1 + \frac{f_{i+1}}{14.f_i} + \frac{f_{i-1}}{14.f_i} + \frac{(f_{i+1} + f_{i-1})^2}{(14.f_i)^2} + \cdots \ldots\ldots\right) \qquad (15)$$

$$y_{i+1} - 2y_i + y_{i-1} = \frac{h^2}{12}(f_{i+1} + 10f_i + f_{i-1}) - \frac{h^2}{588}(13(f_{i+1} + f_{i-1}) - 4f_i)$$

$$+ \frac{12h^2}{(14)^3 f_i}\sum_{0}^{\infty}\frac{(f_{i+1} + f_{i-1})^{m+2}}{(14f_i)^m} \qquad (16)$$

In fact if we truncate second term and onward on right side of above expression (16), we have Numerov method. Thus, from expression (16),we can see Numerov method has insufficient accuracy. Alsowe observe from (16), rational method (13) or (11) is not truncating any term when implementing for computing. So it can be view as difference method with infinite no. of terms to approximate the solution while Numerov method uses finite no. of terms to approximate the solution of differential equation. These facts motivate me to take challenges and observe what happen to different set of problems?

LOCAL TRUNCATION ERROR.

The local truncation error of the proposed difference method (11),as in [11] may be written as:

$$T_i = y_{i+1} - 2y_i + y_{i-1} - \frac{12.h^2(y''_i)^2}{12y''_i - h^2.y_i^{(4)}} \qquad (17)$$

Expanding each term on the right side of (17) in Taylor series about point y_i, so we have:

$$T_i = (h^2 y''_i + \frac{h^4}{12}y_i^{(4)} + \frac{h^6}{360}y_i^{(6)} + o(h^8)) - h^2 y''_i\left(1 - \frac{h^2}{12}\frac{y_i^{(4)}}{y''_i}\right)^{-1} \qquad (18)$$

Assuming that $\left|1 - \frac{h^2}{12}\frac{y_i^{(4)}}{y''_i}\right| < 1$, then by application of binomial theorem on right side of (18) and simplify, we will get:

$$T_i = \frac{h^6}{360}y_i^{(6)} - \frac{h^6}{144}\frac{(y_i^{(4)})^2}{y''_i} + o(h^8)$$

$$T_i = \frac{h^6}{72}\left(\frac{1}{5}y_i^{(6)} - \frac{1}{2}\frac{(y_i^{(4)})^2}{y''_i}\right) + o(h^8) \qquad (19)$$

Thus from (19), the order of the method (11) is four.

NUMERICAL EXPERIMENTS

In this section, the proposed method (11) is applied to solve four different model problems. The resulting system of non-linear equations is solved by Newton–Raphson method. All computations were performed in the GNU FORTRAN language, using double precision. Let y_i, the numerical value calculated by formulae (11) ,an approximate value of the theoretical solution $y(x)$ at the point $x = x_i$. Maximum absolute error,

$$MAE\,(y) = \max_{1 \le i \le N} |y(x_i) - y_i|$$

are shown for different step length h , in Tables 1-4. Also we have shown MAE in the tables, when model problem tested by Numerov Method.

Example 1

Consider the equation,

$$y'' = \frac{2}{x^2}y - \frac{1}{x} \ , \ 2 \le x \le 3 \ ,$$

subject to the boundary condition $y(2) = y(3) = 0$. The exact solution of the problem is

$$y(x) = \frac{1}{38}(19x - 5x^2 - \frac{36}{x}) \ .$$

In Table 1, maximum absolute errors of the present method (11) and classical Numerov method presented.

Example 2

Consider the equation,

$$y''(x) = \frac{\pi^2}{2}((y(x))^4 - (q(x))^4) \ , \ 0 \le x \le 1 \ ,$$

subject to the boundary condition $y(0), y(1)$. Let solution of the problem is

$$y(x) = \sin(\frac{2\pi}{3} + \frac{\pi}{6}) \ .$$

In Table 2, maximum absolute errors of the present method (11) and classical Numerov method presented.

Example 3

Consider the equation,

$$y''(x) = \frac{3}{2}(y(x))^2 \qquad , \qquad 0 \le x \le 1 \ ,$$

subject to the boundary condition $y(0) = 4, y(1) = 1$. Let solution of the problem is

$$y(x) = \frac{4}{(1+x)^2} \ .$$

In Table 3, maximum absolute errors of the present method (11) and classical Numerov method presented.

Example 4

Consider the equation,

$$y''(x) = \frac{1}{(y+1)(y^2+1)} - \frac{y}{(x+1)} + f(x) \ ,$$
$$0 \le x \le 1 \ ,$$

subject to the boundary condition $y(0) = 0 \ , y(1) = \frac{1}{2}$. Let solution of the problem is $yy(x) = \frac{x}{1+x}$.

In Table 4, maximum absolute errors of the present method (11) and classical Numerov method presented.

CONCLUSION

In this article, a non-classical rational method of order four for numerical solution of boundary value problems was described. Stability property is not discussed but truncation error estimated. The decision to use a certain difference scheme does not only depend on the given order of the method but also on its computational efficiency.

The numerical results for model problems show that method (13) is computationally efficient. Also it is observed from the results that method (13) has higher accuracy (i.e., smaller discretization error). The same we cannot say for Numerov Method in considered model problems.

Table 1: Maximum Absolute Error in $y(x) = \frac{1}{38}(19\,x - 5x^2 - \frac{36}{x})$for Example 1.

MAE	N			
	4	8	16	32
(11)	.32028086(-5)	.21231060(-6)	.28487515(-8)	.22935385(-8)
Numerov	.25955862(-5)	.17505769(-6)	.23783494(-7)	.27271618(-7)

Table 2:. Maximum Absolute Error in $y(x) = \sin(\frac{2\pi}{3} + \frac{\pi}{6})$ for Example 2.

MAE	N			
	4	8	16	32
(11)	.14603138(-3)	.92983246(-5)	.29802322(-6)	.59604645(-7)
Numerov	.56743622(-4)	.34570694(-5)	.59604645(-7)	.59604645(-7)

Table 3: Maximum Absolute Error in $y(x) = \frac{1}{1+x}$ for Example 3.

MAE	N			
	4	8	16	32
(11)	.47112084(-2)	.28902054(-3)	.16984939(-4)	.12704658(-6)
Numerov	.23401356(-2)	.15669822(-3)	.91391585(-5)	.12704658(-6)

Table 4: Maximum Absolute Error in $y(x) = \frac{x}{1+x}$ for Example 4.

MAE	N			
	4	8	16	32
(11)	.30243694(-3)	.19767067(-4)	.10288898(-5)	.21559126(-7)
Numerov	.88217857(-4)	.92864038(-5)	.30665021(-6)	.21559126(-7)

Numerical results for four different model examples were given, clearly confirm that the method performs better and faster in linear problem than that of Numerov Method. A numerical result for non-linear model problem is comparable to classical Numerov Method. Our future works will deal with to further improve the accuracy of the method, and to extend this idea to different class of boundary value problems.

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