Crank Nicolson Finite Difference Method for the Valuation of Options.

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ABSTRACT

This paper presents Crank Nicolson finite difference method for the valuation of options. This method attempts to solve the Black Scholes partial differential equation by approximating the differential equation over the area of integration by a system of algebraic equations. It provides a general numerical solution to the valuation problems, as well as an optimal early exercise strategy and other physical sciences. Crank Nicolson method is fairly robust and good for pricing European options.

(Keywords: American option, Crank Nicolson method, European option, finite difference method)

INTRODUCTION

The theory of option pricing continues under construction even though more than thirty years have passed since the publication of the groundbreaking work by Black, Scholes, and Merton. One of the reasons for such a great interest in this subject is the wide range of its applications, which go from financial derivatives to capital budgeting, and more recently, to corporate valuation. In the beginning, options were thought as useful instruments to hedge risk, offering an infinite upside potential and a floor for losses equivalent to the premium or cost of the option. Later on, this concept has been extended to strategic investments under the name of real options. This type of options recognize the flexibility investment decision-makers have to undertake, defer, or abandon an investment, once more information about the project is known.

The Black-Scholes formula is exact when the underlying follows a lognormal distribution. However, in real life, that is not the case and numerical methods shall be used instead. Frequently, asset prices follow non lognormal processes such as stochastic volatility or jump-diffusion ones [3].

A research advance that had improved efficiency and broadened the types of problem where simulation can be applied was described by [4]. Finite difference methods Black Scholes partial differential equation was considered by [5]. The critical factor in option pricing is "the precise description of the stochastic process governing the behavior of the basic asset" and binomial model for approximating the underlying stochastic process directly was considered by [6].

There are many excellent texts and literatures on this subject that may be consulted, such as in references [2], [10], [12], [14], [17], [18], and [19], just to mention few.

In this paper, we shall consider only the accuracy, convergence and the stability of Crank Nicolson finite difference method for the valuation of options that may be exercised only on the expiration date called European options.

FINITE DIFFERENCE METHODS

Many option contract values can be obtained by solving partial differential equations with certain initial and boundary conditions. The finite difference method is one of the premier mathematical tools employed to solve partial differential equations. These methods were pioneered for valuing derivative securities by [5]. The most common finite difference methods for solving the Black-Scholes partial differential equations are the:

- Explicit Method.
- Implicit Method.
- Crank Nicolson method.

These schemes are closely related but differ in stability, accuracy and execution speed, but we shall only consider Crank Nicolson scheme. In the formulation of a partial differential equation problem, there are three components to be considered:

- The partial differential equation.
- The region of space time on which the partial differential is required to be satisfied.
- The ancillary boundary and initial conditions to be met.

Discretization of the Equation

The finite difference method consists of discretizing the partial differential equation and the boundary conditions using a forward or a backward difference approximation. The Black-Scholes partial differential equation is given by:

$$f_t(t,S_t) + rS_t f_{S_t} + \frac{\sigma^2 S^2{}_t}{2} f_{S_tS_t} = rf(t,S_t) \qquad \textbf{(1)}$$

We discretize (1) with respect to time and to the underlying price of the asset. Divide the (t,S_t) plane into a sufficiently dense grid or mesh and approximate the infinitesimal steps Δt and ΔS_t by some small fixed finite steps. Further, define an array of $N+1$ equally spaced grid points $t_0,...,t_N$ to discretize the time derivative with $t_{n+1} - t_n = \frac{T}{N} = \Delta t$. Using the same procedures, we obtain for the underlying price of the asset as follows:

$$S_{M+1} - S_M = \frac{S_{\max}}{M} = \Delta S_t .$$

This gives us a rectangular region on the (t,S_t) plane with sides $(0,S_{\max})$ and $(0,T)$. The grid coordinates (n,m) enables us to compute the solution at discrete points. We will denote the value of the derivative at time step t_n when the underlying asset has value S_m as:

$$f_{m,n} = f(n\Delta t, m\Delta S) = f(t_n, S_m) = f(t, S_t) \qquad \textbf{(2)}$$

where n and m are the numbers of discrete increments in the time to maturity and stock price respectively. The discrete increments in the time to maturity and stock price are given by Δt and ΔS, respectively.

Let $f_n = f_{n,0}.f_{n,1}....f_{n,M}$, $n = 0,1,2,...,N$, then the quantities $f_{0,m}$ .and $f_{N,m}$, for $m = 0,1,2,...,M$ are referred to as the boundary values which may or may not be known ahead of time but in our partial differential equation they are known. The quantities $f_{n,m}$ for $n = 1,2,...,N-1$ and $m = 0,1,2,...,M$ are referred to as interior points or values.

Boundary and Initial Conditions

Partial differential equations can be classified as:

- Boundary value problems, where we need to specify the full set of boundary conditions.
- Initial value problems, where only the values of the function at one particular time needs to be specified. The majority of derivative security pricing problems, including most of the options valuation problems are initial problems.

Partial differential equations without the ancillary boundary or initial conditions will either have an infinitely many solutions or has no solution. We need to specify the boundary and initial conditions for the call and put options whose payoff are given by $(S_T - K)^+$ and $(K - S_T)^+$ respectively, where S_T is the stock price at time T and K is the exercise or strike price.

When the price is worth nothing, a put is worth its strike price, i.e.,

$$f_{n,0} = K, n = 0,1,2,...,N .$$

As the price of the underlying asset increases, the value of the put option approaches zero.

Accordingly, we choose $S_{\max} = S_m$ and from this we get:

$$f_{n,M} = 0, n = 0,1,2,...,N .$$

We know that the value of the European put option at time T and can impose the initial condition:

$$f_{N,m} = (K - m\Delta S_T)^+, m = 0,1,2,...,M .$$

The initial condition gives us the value of f at the end of the time period and not at the beginning. This means that we move backward from the maturity date to time zero. The price of the put option is given by $f_{0,\frac{M+1}{2}}$ (odd) and $f_{0,\frac{M}{2}}$ (even).

This method is suited for European put option where early exercise is not permitted. The call-put parity given by:

$$C_E + Ke^{-rt} = P_E + S$$

can be used to obtain the corresponding value of European call option. To value put option, where early exercise is permitted, we need to make only one simple modification [16, 20]. After each linear system solution, we need to consider whether early exercise is optimal or not. We compare the option value with the intrinsic value of the option. If the intrinsic value is greater, then set option value to the intrinsic value.

The American call and put options are handled in almost exactly the same way. We have for the call and put options respectively:

$$\left. \begin{aligned} f_{N,m} &= (m\Delta S_T - K)^+ \\ f_{N,m} &= (K - m\Delta S_T)^+, m = 0,1,2,...,M \end{aligned} \right\} \quad \textbf{(3)}$$

Finite Difference Approximations

In finite difference method, we replace the partial derivative occurring in the partial differential equation by approximations based on Taylor series expansions of function near the points of interest [15]. The derivative we seek is expressed with many desired order of accuracy.

Assuming that $f(t,S)$ represented in the grid by $f_{n,m}$ [1], the respective expansions of $f(t,\Delta S + S)$ and $f(t,S-\Delta S)$ in Taylor series are given by equations:

$$f(t,S_t+\Delta S_t) = f(t,s_t) + f_s\Delta S_t + \frac{1}{2} f_{ss}\Delta^2 S_t + \frac{1}{6} f_{sss}\Delta^3 S_t + O(\Delta^4 S_t) \quad \textbf{(4)}$$

$$f(t,S_t-\Delta S_t) = f(t,s_t) - f_s\Delta S_t + \frac{1}{2} f_{ss}\Delta^2 S_t - \frac{1}{6} f_{sss}\Delta^3 S_t + O(\Delta^4 S_t) \quad \textbf{(5)}$$

Using equation (4), the forward difference is given by:

$$f_{S_t} \approx \frac{f_{n,m+1} - f_{n,m}}{\Delta S_t} \quad \textbf{(6)}$$

Where:

$f(t,S_t+\Delta S_t) = f_{n,m+1}, f(t,S_t-\Delta S_t) = f_{n,m-1}$ and $f(t,S) = f_{n,m}$

Equation (5) gives the corresponding backward difference as:

$$f_{S_t} \approx \frac{f_{n,m} - f_{n,m-1}}{\Delta S_t} \quad \textbf{(7)}$$

Subtracting (5) from (4) and taking the first order partial derivative, we have the central difference given by:

$$f_{S_t} \approx \frac{f_{n,m+1} - f_{n,m-1}}{2\Delta S_t} \quad \textbf{(8)}$$

The second order partial derivative can be estimated by the symmetric central difference approximation. Adding Equations (4) and (5) and taking the second order partial derivative we have:

$$f_{S_tS_t} \approx \frac{f_{n,m+1} - 2f_{n,m} - f_{n,m-1}}{\Delta S_t^{\ 2}} \quad \textbf{(9)}$$

Although there are other approximations, this approximation to (9) is preferred; it is also invariant and more accurate than the similar approximations [1].

We expand $f(t+\Delta t, S)$ in Taylor series as:

$$f(t+\Delta t,S) = f(t,s_t) + f_t\Delta t + \frac{1}{2}f_{tt}\Delta^2 t + \frac{1}{6}f_{ttt}\Delta^3 t + O(\Delta^4 t) \quad \textbf{(10)}$$

The forward difference for the maturity time given by:

$$f_t \approx \frac{f_{n+1,m} - f_{n,m}}{\Delta S_t} \quad \textbf{(11)}$$

Substituting Equations (8), (9), and (11) into (1), we have:

$$\rho_{1m} f_{n,m-1} + \rho_{2m} f_{n,m} + \rho_{3m} f_{n,m+1} = f_{n+1,m} \quad \textbf{(12)}$$

Where,

$$\begin{aligned} v_{1m} &= \frac{rm\Delta t}{2} - \frac{\sigma^2 m^2 \Delta t}{2} \\ v_{2m} &= 1 + r\Delta t + \sigma^2 m^2 \Delta t \\ \rho_{3m} &= -\frac{rm\Delta t}{2} - \frac{\sigma^2 m^2 \Delta t}{2} \end{aligned} \quad \textbf{(13)}$$

Equation (11) is called a finite difference equation which gives equation that we use to approximate the solution of $f(t,S)$ [4, 9].

Similarly, we obtained for the explicit and implicit finite difference method as follows [13]:

For explicit case:

$$\frac{1}{1+r\Delta t}(\alpha_{1m} f_{n+1,m-1} + \alpha_{2m} f_{n+1,m} + \alpha_{3m} f_{n+1,m+1}) = f_{n,m} \quad \textbf{(14)}$$

Where,

$$\begin{aligned} \alpha_{1m} &= \frac{\sigma^2 m^2 \Delta t}{2} - \frac{rm\Delta t}{2} \\ \alpha_{2m} &= 1 - \sigma^2 m^2 \Delta t \\ \alpha_{3m} &= \frac{\sigma^2 m^2 \Delta t}{2} + \frac{rm\Delta t}{2} \end{aligned} \quad \textbf{(15)}$$

This method is accurate up to $O(\Delta t, \Delta^2 S_t)$. The problem associated with the explicit method is that some probabilities are negative. This produces results that do not converge to the solution of the differential equation. The condition for the method to have non-negative probabilities is that $\sigma^2 m^2 \Delta t$ and $r < \sigma^2 m$ [11].

For implicit case we have,

$$\frac{1}{1-r\Delta t}(\beta_{1m} f_{n,m-1} + \beta_{2m} f_{n,m} + \beta_{3m} f_{n,m+1}) = f_{n+1,m} \quad \textbf{(16)}$$

Then the parameters in (16) are given by:

$$\begin{aligned} \beta_{1m} &= -\frac{\sigma^2 m^2 \Delta t}{2} + \frac{rm\Delta t}{2} \\ \beta_{2m} &= 1 + \sigma^2 m^2 \Delta t \\ \beta_{3m} &= -\frac{\sigma^2 m^2 \Delta t}{2} - \frac{rm\Delta t}{2} \end{aligned} \quad \textbf{(17)}$$

Similar to the explicit method, implicit method has accuracy up to $O(\Delta t, \Delta^2 S_t)$.

Crank Nicolson Finite Difference Method

The Crank Nicolson Method is the average of the explicit and implicit methods. The explicit and implicit methods are given by Equations (14) and (16), respectively. We than take the average of the two methods to get:

$$\begin{aligned} &\frac{1}{1-r\Delta t}(\beta_{1m} f_{n,m-1} + \beta_{2m} f_{n,m} + \beta_{3m} f_{n,m+1}) + \\ &\frac{1}{1+r\Delta t}(\alpha_{1m} f_{n+1,m-1} + \alpha_{2m} f_{n+1,m} + \alpha_{3m} f_{n+1,m+1}) = f_{n,m} + f_{n+1,m} \end{aligned} \quad \textbf{(18)}$$

Substituting (15) and (17) into (18), we have:

$$\left(\frac{rm\Delta t}{4}-\frac{\sigma^2 m^2\Delta t}{4}\right)f_{n,m-1}+\left(1+\frac{r\Delta t}{2}+\frac{\sigma^2 m^2\Delta t}{2}\right)f_{n,m}$$
$$+\left(-\frac{rm\Delta t}{4}-\frac{\sigma^2 m^2\Delta t}{4}\right)f_{n,m+1}=\left(\frac{\sigma^2 m^2\Delta t}{4}-\frac{rm\Delta t}{4}\right)f_{n+1,m-1}$$
$$+\left(1-\frac{r\Delta t}{2}-\frac{\sigma^2 m^2\Delta t}{2}\right)f_{n+1,m}+\left(\frac{rm\Delta t}{4}+\frac{\sigma^2 m^2\Delta t}{4}\right)f_{n,m}$$

Then we have

$$\nu_{1m} f_{n,m-1}+\nu_{2m} f_{n,m}+\nu_{3m} f_{n,m+1}=\varphi_{1m} f_{n+1,m+1}$$
$$+\varphi_{2m} f_{n+1,m}+\varphi_{3m} f_{n+1,m+1} \qquad \textbf{(18b)}$$

Where the parameters ν_{km} and φ_{km} for $k=1,2,3$ are given by:

$$\nu_{1m}=\frac{rm\Delta t}{4}-\frac{\sigma^2 m^2\Delta t}{4}$$
$$\nu_{2m}=1+\frac{r\Delta t}{2}+\frac{\sigma^2 m^2\Delta t}{2}$$
$$\nu_{3m}=-\frac{rm\Delta t}{4}-\frac{\sigma^2 m^2\Delta t}{4}$$
$$\varphi_{1m}=-\frac{rm\Delta t}{4}+\frac{\sigma^2 m^2\Delta t}{4} \qquad \textbf{(19)}$$
$$\varphi_{2m}=1-\frac{r\Delta t}{2}-\frac{\sigma^2 m^2\Delta t}{2}$$
$$\varphi_{3m}=\frac{rm\Delta t}{4}+\frac{\sigma^2 m^2\Delta t}{4}$$

$n=0,1,2,...,N-1$ and $m=1,2,...,M-1$ [9]. Equation (18) is called Crank Nicolson finite difference method.

<u>The Accuracy of Crank Nicolson Finite Difference Method</u>

The finite difference approximations from the Taylor series expansion lead to truncation errors and this affects the accuracy of the scheme [12]. This method is more accurate than the explicit and implicit methods with accuracy up to $O(\Delta^2 t,\Delta^2 S_t)$. This accuracy can be shown by equating the central difference and the symmetric central difference at:

$$f_{n+\frac{1}{2},m}\approx f\left(t+\frac{\Delta t}{2},S_T\right)$$

We expand $f_{n+1,m}$ and $f_{n,m}$ 1in Taylor series at $f_{n+\frac{1}{2},m}$ to yield $f_{n+1,m}=f_{n+\frac{1}{2},m}+\frac{1}{2}f_t\Delta t+O(\Delta^2 t)$ and $f_{n,m}=f_{n+\frac{1}{2},m}-\frac{1}{2}f_t\Delta t+O(\Delta^2 t)$, respectively. Taking the average of these equations, we have:

$$\frac{1}{2}\left(f_{n,m}+f_{n+1,m}\right)=f_{n+\frac{1}{2},m}+O(\Delta^2 t)$$

The subscript m in the last equation above is arbitrary. Then we can write for $m-1,m$ and $m+1$ as follows:

$$f_{n+\frac{1}{2},m-1}-2f_{n+\frac{1}{2},m}+f_{n+\frac{1}{2},m+1}=$$
$$\frac{1}{2}\left(f_{n,m-1}-2f_{n,m}+f_{n,m+1}\right) \qquad \textbf{(20)}$$
$$+\frac{1}{2}\left(f_{n+1,m-1}-2f_{n+1,m}+f_{n+1,m+1}\right)+O(\Delta^2 t)$$

The right hand side of (20) is an average of two symmetric central differences centered at grid points n and $n+1$. Dividing by $\Delta^2 S_T$ we obtain the equality:

$$\frac{\partial^2 f(t+\frac{1}{2}\Delta t,S_T)}{\partial S_T^2}=\frac{1}{2}\left(\frac{\partial^2 f(t,S_T)}{\partial S_T^2}+\frac{\partial^2 f(t+\Delta t,S_T)}{\partial S_T^2}\right)$$
$$+O(\Delta^2 t,\Delta^2 S_T)$$

Which is the second order derivative defined by the symmetric central difference approximation. The subscript m is arbitrary and we derive the central difference approximation as follows:

$$f_{n+\frac{1}{2},m+1} - f_{n+\frac{1}{2},m-1} = \frac{1}{2}\left(f_{n,m+1} - f_{n,m}\right) + \frac{1}{2}\left(f_{n+1,m+1} - f_{n+1,m+1}\right) + O(\Delta^2 t) \quad \textbf{(21)}$$

We divide the above equation (21) by $2\Delta S_T$ to get the equality of the form:

$$\frac{\partial f(t+\frac{1}{2}\Delta t, S_T)}{\partial S} = \frac{1}{2}\left(\frac{\partial f(t,S_T)}{\partial S_T} + \frac{\partial f(t+\Delta t, S_T)}{\partial S_T}\right) + O(\Delta^2 t, \Delta^2 S_T)$$

Which is the first order partial derivative defined by the symmetric central difference approximation. Subtracting $f_{n,m}$ from $f_{n+1,m}$, we have the approximation of $\frac{\partial f}{\partial t}$ centered at $\left(t+\frac{1}{2}\Delta t, S_T\right)$ given by:

$$\frac{\partial f\left(t+\frac{1}{2}\Delta t, S_T\right)}{\partial t} = \frac{f_{n+1,m} - f_{n,m}}{\Delta t} + O(\Delta^2 t)$$

Hence, the Black Scholes partial differential equation centered at $\left(t+\frac{1}{2}\Delta t, S_T\right)$ has a finite difference approximation.

$$\left(\frac{rm\Delta t}{4} - \frac{\sigma^2 m^2 \Delta t}{4}\right) f_{n,m-1} + \left(1 + \frac{r\Delta t}{2} + \frac{\sigma^2 m^2 \Delta t}{2}\right) f_{n,m} + \left(-\frac{rm\Delta t}{4} - \frac{\sigma^2 m^2 \Delta t}{4}\right) f_{n,m+1} = \left(\frac{\sigma^2 m^2 \Delta t}{4} - \frac{rm\Delta t}{4}\right) f_{n+1,m-1} + \left(1 - \frac{r\Delta t}{2} - \frac{\sigma^2 m^2 \Delta t}{2}\right) f_{n+1,m} + \left(\frac{rm\Delta t}{4} + \frac{\sigma^2 m^2 \Delta t}{4}\right) f_{n,m}$$

Re – writing the above equation as:

$$v_{1m} f_{n,m-1} + v_{2m} f_{n,m} + v_{3m} f_{n,m+1} = \varphi_{1m} f_{n+1,m+1} + \varphi_{2m} f_{n+1,m} + \varphi_{3m} f_{n+1,m+1}$$

We get (18) which is the exact Crank Nicolson Method. Therefore, this method has a leading error of order $(\Delta^2 t, \Delta^2 S_T)$.

Stability Analysis [7, 8]

The two fundamental sources of error are the truncation error in the stock price discretization and in the time discretization. The importance of truncation error is that the numerical scheme solves a problem that is not exactly the same as the problem we are trying to solve.

The three fundamental factors that characterize a numerical scheme are consistency, stability and convergence [9].

- Consistency: A finite difference of a partial differential equation is consistent, if the difference between partial differential equation and finite differential equation vanishes as the interval and time step size approach zero, i.e. as $n \to \infty, (PDE - FDE) \to 0$

  Consistency deals with how well the finite difference equation approximates the partial differential equation and it is the necessary condition for convergence.

- Stability: For a stable numerical scheme, the errors from any source will not grow unboundedly with time.

- Convergence: It means that the solution to a finite difference equation approaches the true solution to the partial differential equation as both grid interval and time step sizes are reduced. The necessary and sufficient conditions for convergence are consistency and stability.

These three factors that characterize a numerical scheme are linked together by Lax-Richtmyer equivalence theorem [8, 9]

The Lax-Richtmyer Equivalence Theorem

The Lax-Richtmyer Equivalence Theorem is often called the *Fundamental Theorem of Numerical*

Analysis even though it is only applicable to the small subset of linear numerical methods for well posed, linear differential equations. Along with Dahlquist's equivalence theorem for ordinary differential equations, the notion that the relationship *Consistency* + *Stability* $\Leftrightarrow$ *Convergence* always holds has caused a great deal of confusion in the numerical analysis of differential equations. In the case of partial differential equations, mathematicians are most often interested in nonlinear phenomena, for which Lax-Richtmyer does not apply.

More damningly, the forward implication that *Consistency* + *Stability* $\Rightarrow$ *Convergence* is trivial for linear schemes, and thus it is only the converse notion that *convergence* $\Rightarrow$ *stability* that the theorem contributes. The intuition that the theorem gives for problems that fall outside the scope of Lax-Richtmyer, however, is fairly, since consistency and stability are often insufficient for convergence, and convergence need not imply stability in general.

Lax-Richtmyer theorem [15] states that given a well posed linear initial value problem and a consistent finite difference scheme (positive order of accuracy), stability is the necessary and sufficient condition for convergence. In general, a problem is said to be well posed if:

- A solution to the problem exists.
- The solution is unique when it exists.
- The solution depends continuously on the problem data.

A Necessary and Sufficient Condition for Stability

Let $f_{n+1} = Af_n$ be a system of equations, where A and f_{n+1} are matrix and column vectors respectively [7]. Then:

$$\begin{aligned} f_n &= Af_{n-1} \\ &= A^2 f_{n-2} \\ &= A^3 f_{n-3} \\ &\vdots \\ &= A^{n-1} f_1 \\ &= A^n f_0 \end{aligned} \quad \textbf{(22)}$$

For $n = 1,2,...,N$ and f_0 is the vector of initial value. We are concerned with stability and we also perturbed the vector of the initial value f_0 to h_0. The exact solution at the n^{th} row will then be:

$$h_n = A^n h_0 \quad \textbf{(23)}$$

Let the perturbation or error vector e be denoted by $e = h - f$ and using the perturbation vectors (22) and (23), we have:

$$\begin{aligned} e_n &= h_n - f_n \\ &= A^n f_0 - A^n h_0 \\ &= A^n (h_0 - f_0) \end{aligned}$$

Therefore,

$$e_n = A^n e_0$$

Hence for compatible matrix and vector norms [15]

$$\|e_n\| \le \|A^n\| \, \|e_0\|$$

Lax and Richtmyer defined the difference scheme to be stable when there exists a positive number L which is independent of n, Δt and ΔS_T such that:

$$\|A^n\| \le L, n = 1,2,.., N .$$

This limits the amplification of any initial perturbation and therefore of any arbitrary initial rounding errors [8], i.e.,

$$\|e_n\| \le L \|e_0\|$$

Since,

$\|A^n\| \le \|A^{n-1}A\| \le \|A\| \|A^{n-1}\| \le ... \|A\|^n$, then the Lax-Richmyer definition of stability is satisfied when:

$$\|A\| \le 1 \quad \textbf{(24)}$$

Hence (24) is the necessary and sufficient condition for finite difference equations to be stable [9]. Since the spectral radius $\rho(A)$ satisfies $\rho(A) \leq \| A \|$, it follows from (24) that $\rho(A) \leq 1$. We note that if matrix A is real and symmetric, then by [9] we have:

$$\|A\|_{\infty} = \max|\lambda_n|$$

The Eigenvalues of a Common Tridiagonal Matrix

The other method used in the analysis of stability is the eigenvalue of the tridiagonal system. The eigenvalues of the tridiagonal matrix are given by:

$$\lambda_n = y + 2(\sqrt{xz})\cos\frac{n\pi}{N}, n = 1,2,...,N, x, y, z \in R \qquad \textbf{(25)}$$

By Lax equivalence theorem, the three finite difference methods are consistent and convergent but in the analysis of their stability, explicit method is quite stable, the implicit method is conditionally stable and the Crank Nicolson method is unconditionally stable finite difference method because it calculates small change in the option value for a small change in the initial conditions, converges to the solution of the partial differential equation and calculation error decreases when number of time and price partitions increase.

NUMERICAL EXAMPLES AND RESULTS

Example 1

Consider a standard option that expires in six months with a strike price of $55. Assume that the underlying stock pays no dividend, trades at $60 and has a volatility of 25% per annum. The risk-free rate is 10% per annum.

We compute the values of European call and put options using Crank Nicolson method as we increase the number of steps M and N with the following parameters:

$$S = \$60, K = \$55, r = 0.1, \sigma = 0.25, T = 0.5$$

The Black-Scholes price for the call and put options are $8.9234 and $1.2410, respectively. The results obtained are shown in the Tables 1 and 2.

Table 1: The Results of Crank Nicolson Method as we increase M and N.

$M = N$	European Call	European Put
10	9.0514	1.2596
20	8.8555	1.1429
30	8.9394	1.2413
40	8.9090	1.2168
50	8.9300	1.2411
60	8.9172	1.2303
70	8.9265	1.2410
80	8.9194	1.2350
90	8.9246	1.2410
100	8.9201	1.2371

Table 2: Illustrative Results for the Performance of the Crank Nicolson Method when M and N are Different.

M	N	European Call	European Put
20	10	8.8558	1.1434
40	20	8.9091	1.2170
60	30	8.9173	1.2304
80	40	8.9194	1.2350
100	50	8.9202	1.2372
120	60	8.9204	1.2390
140	70	8.9204	1.2383
160	80	8.9203	1.2395
180	90	8.9203	1.2398
200	100	8.9202	1.2400

Example 2

We price the European call option on a non-dividend paying stock with the following parameters:

$$S = \$50, K = \$60, r = 0.05, \sigma = 0.2, T = 1$$

The Black-Scholes price for the call option is 1.6237. The results obtained is shown in the Table 3 below and the illustrative result for the performance of the Crank Nicolson method when N and M are different is shown in Table 4 below.

Table 3: The Results of Crank Nicolson Method as we increase M and N.

$M = N$	Crank Nicolson Method
10	1.4782
20	1.5739
30	1.6010
40	1.6110
50	1.6156
60	1.6181
70	1.6196
80	1.6205
90	1.6212
100	1.6216

Table 4: Illustrative Results for the Performance of the Crank Nicolson Method when M and N are Different.

M	N	Crank Nicolson Method
20	10	1.5731
40	20	1.6108
60	30	1.6180
80	40	1.6205
100	50	1.6216
120	60	1.6222
140	70	1.6225
160	80	1.6227
180	90	1.6229
200	100	1.6230

Example 3

We consider the performance of the Crank Nicolson method against the analytic Black-Scholes price for a European put option with the following parameters:

$$K = \$50, r = 0.05, \sigma = 0.25, T = 3.0$$

The results obtained are shown in the Table 5 below.

DISCUSSION OF RESULTS

Tables 1 and 3 show that the Crank Nicolson method performs well, is consistent as $M \to \infty,\ \Delta S_T \to 0\ N \to \infty,\ \ \Delta t \to 0$.

Table 5: A Comparison of Crank Nicolson Method with the Black-Scholes Price for a European Put Option.

Stock Price S.	Black-Scholes Analytic Price	Crank Nicolson Method
10	33.0363	33.0362
15	28.0619	28.0616
20	23.2276	23.2271
25	18.7361	18.7350
30	14.7739	14.7734
35	11.4384	11.4390
40	8.7338	8.7334
45	6.6021	6.6019
50	4.9564	4.9563
55	3.7046	3.7042
60	2.7621	2.7613
65	2.0574	2.7613
70	1.5328	1.5326
75	1.1430	1.1427
80	0.8538	0.8537
85	0.6392	0.6391
90	0.4797	0.4795
95	0.2501	0.2490
100	0.2319	0.2315

Tables 2 and 4 show that when N and M are different, (i.e. the number of time steps N initially set at 10 and doubled with each grid M refinement), the Crank Nicolson finite difference method converges faster than when the number of steps N and M are the same.

Table 5 shows the variation of the option price with the underlying price S. The results demonstrate that Crank Nicolson method is good for pricing European option, accurate and agree with the Black-Scholes value.

CONCLUSION

In general, each numerical method has its advantages and disadvantages of use: Crank Nicolson finite difference method converges faster and more accurate, it is fairly robust and good for pricing European put and call options. Crank Nicolson method requires sophisticated algorithms for solving large sparse linear systems of equations, somewhat problematic for path dependent options and is relatively difficult to code.

Finally, we conclude that Crank Nicolson method is unconditionally stable, convergent and more accurate when pricing European option.

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