On the Number of Conjugacy Classes in the Injective Order-Decreasing Transformation Semigroup.

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ABSTRACT

One of the problems we face in transformation semigroups is its enumeration (Umar, 2010). The injective order-decreasing transformation semigroup is a sub-semigroup of injective transformation semigroup. Its elements were studied and arranged in their respective conjugacy class while applying the use of two kinds of path structure (the Circuit and Proper path) in its path decomposition. The conjugacy classes were also arranged according to the number of its images in any number of the Injective order-decreasing transformation Semigroup. A general expression was obtained for the number of conjugacy classes in the order-decreasing transformation semigroup.

(Keywords: conjugacy classes, transformation semigroup, injective order-decreasing transformation semigroup)

PRELIMINARIES

A transformation semigroup is a pair (X, S), where X is a set and S is a semigroup of transformations of X. Here, a transformations of X is a just a function from X to itself, not necessarily with an inverse, and therefore S is simply a set of transformations of X which is closed under composition of functions.

Let $X_n = \{1, 2, \ldots, n\}$. Then a (partial) transformation $\alpha: Domain\ \alpha \subseteq X_n \to Im\alpha$ is said to be full or total if $Dom\ \alpha = X_n$, otherwise it is called strictly partial. The set of all partial transformation on n-object forms a semigroup under the usual composition of functions. It is denoted by P_n, when it is strictly partial, T_n when it is full or total and I_n when it is partial one-one.

A transformation $\alpha\ in\ I_n$ is said to be order-decreasing if $(\forall\ x \in Dom\ \alpha) x\alpha \leq x$. The semigroup of order-decreasing partial one-one is denoted as ID_n. (Umar, 2010).

CONJUGACY

If G is a group, a conjugacy relation on G is an equivalence relation on G. If $a, b \in G$ and there exist $g \in G$, a is conjugate to b if $b = gag^{-1}$.

In conjugacy classes of permutation group S_n, it can be represented using a cycle type which also shows the partition of n.

In semigroups, conjugacy classes are represented using path structure of which there are two kinds, the proper path and the circuit. Let $a_1, a_2, \ldots, a_n \in N$ be the domain and α as the function mapping each element of the domain to its image such that $\alpha(a_1) = a_2, \alpha(a_2) = a_3, \ldots, \alpha(a_{n-1}) = a_n\ and\ \alpha(a_n) = p$.

If $p = a_1$, then we have a circuit path otherwise it is known as a proper path that is when $p \neq a_1$.

METHODOLOGY

In the path decomposition of the semigroup, two kinds of path structure was applied, the Circuit and Proper path.

Let $a_1, a_2, \dots, a_n \in N$ be the domain and α as the function mapping each element of the domain to its image such that: $\alpha(a_1) = a_2, \alpha(a_2) = a_3, \dots, \alpha(a_{n-1}) = a_n$ and $\alpha(a_n) = p$.

If $p = a_1$, then we have a circuit path otherwise it is known as a proper path that is when $p \neq a_1$.

For instance, the path decomposition of $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & - & 4 & - & 7 & 8 & 9 & 6 \end{pmatrix} \in I_9$ is given as (123](4)(5](6789)

Extending the use of cycle or path structure to transformation semigroups we have;

Theorem 1

Let $\alpha, \beta \in I_n$, then chart α is conjugate to chart β if and only if α and β have the same path structure (See proof in Lipscomb, 1995, pp 10-11).

For instance, the size of the conjugacy class of (1)(2)(3](4](5] $\in ID_5$ is 10: (1)(2)(3](4](5],

(1)(2](3)(4](5], (1)(2](3](4)(5], (1)(2](3](4](5), (1](2)(3)(4](5], (1](2)(3](4)(5],

(1](2)(3](4](5), (1](2](3)(4)(5], (1](2](3)(4](5), (1](2](3](4)(5)

CONJUGACY CLASSES IN THE INJECTIVE ORDER-DECREASING TRANSFORMATION SEMIGROUP

The conjugacy classes are arranged according to the number of their images in any number of ID_n.

When $n = 1 : ID_1$

Number of Image	Conjugacy classes
No Image	(1]
1 Image	(1)

Total Conjugacy classes = 2

When $n = 2 : ID_2$

Number of Image	Conjugacy classes
No Image	(1](2]
1Image	(1)(2], (21]
2 Images	(1)(2)

Total Conjugacy classes = 4

When $n = 3 : ID_3$

Number of Image	Conjugacy classes
No Image	(1](2](3]
1 Image	(1)(2](3], (1](32]
2 Images	(1)(2)(3], (1)(32], (321]
3 Images	(1)(2)(3)

Total Conjugacy classes = 7

When $n = 4 : ID_4$

Number of Image	Conjugacy classes
No Image	(1](2](3](4]
1 Image	(1)(2](3](4], (21](3](4]
2 Images	(1)(2)(3](4], (1)(2](43], (1](432], (21](43]
3 Images	(1)(2)(3)(4], (1)(2)(43], (1)(432], (4321]
4 Images	(1)(2)(3)(4)

Total Conjugacy classes = 12

When $n = 5: ID_5$

Number of Image	Conjugacy classes
No Image	1][2](3](4](5]
1 Image	(1)(2](3](4](5], (1](2](3](54]
2 Images	(1)(2)(3](4](5], (1)(2](3](54], (1](2](543], (1](32](54]
3 Images	(1)(2)(3)(4](5], (1)(2)(3](54], (1)(2](543], (1)(32](54], (1](5432], (21](543]
4 Images	(1)(2)(3)(4)(5], (1)(2)(3)(54], (1)(2)(543], (1)(5432], (54321]
5 Images	(1)(2)(3)(4)(5)

Total Conjugacy classes = 19

When $n = 6: ID_6$

Number of Image	Conjugacy classes
No Image	(1](2](3](4](5](6]
1 Image	(1)(2](3](4](5](6], (1](2](3](4](65]
2 Images	(1)(2)(3](4](5](6], (1)(2](3](4](65], (1](2](3](654], (1](2](43](65]
3 Images	(1)(2)(3)(4](5](6], (1)(2)(3](4](65], (1)(2](3](654], (1)(2](43](65], (1](2](6543], (1](32](654], (21](43](65]
4 Images	(1)(2)(3)(4)(5](6], (1)(2)(3)(4](65], (1)(2)(3](654], (1)(2)(43](65], (1)(2](6543], (1)(32](654], (1](65432], (21](6543], (321](654]
5 Images	(1)(2)(3)(4)(5)(6], (1)(2)(3)(4)(65], (1)(2)(3)(654], (1)(2)(6543], (1)(65432], (654321]
6 Images	(1)(2)(3)(4)(5)(6)

Total Conjugacy classes = 30

When $n = 7: ID_7$

Number of Image	Conjugacy classes
No Image	(1](2](3](4](5](6](7]
1 Image	(1)(2](3](4](5](6](7], (1](2](3](4](5](76]
2 Images	(1)(2)(3](4](5](6](7], (1)(2](3](4](5](76], (1](2](3](4](765], (1](2](3](45](76]
3 Images	(1)(2)(3)(4](5](6](7], (1)(2)(3](4](5](76], (1)(2](3](4](765], (1)(2](3](54](76], (1](2](3](7654], (1](2](43](765], (1](32](54](76]
4 Images	(1)(2)(3)(4)(5](6](7], (1)(2)(3)(4](5](76], (1)(2)(3](4](765], (1)(2)(3](54](76], (1)(2](3](7654], (1)(2](43](765], (1)(32](54](76], (1](2](76543], (1](32](7654], (1](432](765], (21](43](765]
5 Images	(1)(2)(3)(4)(5)(6](7], (1)(2)(3)(4)(5](76], (1)(2)(3)(4](765], (1)(2)(3)(54](76], (1)(2)(3](7654], (1)(2)(43](765], (1)(2](76543], (1)(32](7654], (1)(432](765], (1](765432], (21](76543], (321](7654]
6 Images	(1)(2)(3)(4)(5)(6)(7], (1)(2)(3)(4)(5)(76], (1)(2)(3)(4)(765], (1)(2)(3)(7654], (1)(2) (76543], (1)(765432], (7654321]
7 Images	(1)(2)(3)(4)(5)(6)(7)

Total Conjugacy classes = 45

When $n = 8: ID_8$

Number of Image	Conjugacy classes
No Image	(1](2](3](4](5](6](7](8]
1 Image	(1)(2](3](4](5](6](7](8], (1](2](3](4](5](6](87]
2 Images	(1)(2)(3](4](5](6](7](8], (1)(2](3](4](5](6](87], (1](2](3](4](5](876], (1](2](3](4](65](87]
3 Images	(1)(2)(3)(4](5](6](7](8], (1)(2)(3](4](5](6](87], (1)(2](3](4](5](876], (1)(2](3](4](56](87], (1](2](3](4](8765], (1](2](3](54](876], (1](2](43](65](87]
4 Images	(1)(2)(3)(4)(5](6](7](8], (1)(2)(3)(4](5](6](87], (1)(2)(3](4](5](876], (1)(2)(3](4](65](87], (1)(2](3](4](8765], (1)(2](3](54](876], (1)(2](43](65](87], (1](2](3](87654], (1](2](43](8765], (1](2](543](876], (1](32](54](876], (21](43](65](87]
5 Images	(1)(2)(3)(4)(5)(6](7](8], (1)(2)(3)(4)(5](6](87], (1)(2)(3)(4](5](876], (1)(2)(3)(4](65](87], (1)(2)(3](4](8765], (1)(2)(3](54](876], (1)(2)(43](65](87], (1)(2](3](87654], (1)(2](43](8765], (1)(2](543](876], (1)(32](54](876], (1](2](876543], (1](32](87654], (1](432](8765], (21](43](8765], (21](543](876]

6 Images	(1)(2)(3)(4)(5)(6)(7][8], (1)(2)(3)(4)(5)(6][87], (1)(2)(3)(4)(5][876], (1)(2)(3)(4)(65][87], (1)(2)(3)(4][8765], (1)(2)(3)(54][876], (1)(2)(3][87654], (1)(2)(43][8765], (1)(2)(543][876], (1)(2][876543], (1)(32][87654], (1)(432][8765], (1][8765432], (21][876543], (321][87654], (4321][8765],
7 Images	(1)(2)(3)(4)(5)(6)(7)(8], (1)(2)(3)(4)(5)(6)(87], (1)(2)(3)(4)(5)(876], (1)(2)(3)(4)(8765], (1)(2)(3)(87654], (1)(2)(876543], (1)(8765432], (87654321]
8 Images	(1)(2)(3)(4)(5)(6)(7)(8)

Total conjugacy classes = 67

When $n = 9 : ID_9$

Number of Image	Conjugacy classes
No Image	(1][2][3][4][5][6][7][8][9]
1 Image	(1)(2][3][4][5][6][7][8][9], (1][2][3][4][5][6][7][98]
2 Images	(1)(2)(3][4][5][6][7][8][9], (1)(2][3][4][5][6][7][98], (1][2][3][4][5][6](987], (1][2][3][4][5](76](98]
3 Images	(1)(2)(3)(4][5][6][7][8][9], (1)(2)(3][4][5][6][7](98], (1)(2][3][4][4][5][6](987], (1)(2][3][4][5](76](98], (1][2][3][4][5](9876], (1][2][3][4](65](987], (1][2][3](54](76](98],
4 Images	(1)(2)(3)(4)(5][6][7][8][9], (1)(2)(3)(4][5][6][7](98], (1)(2)(3][4][5][6](987], (1)(2)(3][4][5](76](98], (1)(2][3][4][5](9876], (1)(2][3][4](65](987], (1)(2][3](54](76](98], (1][2][3][4](98765], (1][2][3](54](9876], (1][2][3](654](987], (1][2](43](65](987], (1](32](54](76](98],
5 Images	(1)(2)(3)(4)(5)(6][7][8][9], (1)(2)(3)(4)(5][6][7](98], (1)(2)(3)(4][5][6](987], (1)(2)(3)(4][5](76](98], (1)(2)(3][4][5](9876], (1)(2)(3][4](65](987], (1)(2)(3](54](76](98], (1)(2][3][4](98765], (1)(2][3](54](9876], (1)(2][3](654](987], (1)(2](43](65](987], (1)(32](54](76](98], (1][2][3](987654], (1][2](43](98765], (1][2](543](9876], (1](32](54](9876], (1](32](654](987], (21](43](65](987]
6 Images	(1)(2)(3)(4)(5)(6)(7][8][9], (1)(2)(3)(4)(5)(6][7](98], (1)(2)(3)(4)(5][6](987], (1)(2)(3)(4)(5](76](98], (1)(2)(3)(4][5](9876], (1)(2)(3)(4](65](987], (1)(2)(3)(45](76](98], (1)(2)(3][4](98765], (1)(2)(3](54](9876], (1)(2)(3](654](987], (1)(2)(43](65](987], (1)(2][3](987654], (1)(2](43](98765], (1)(2](543](9876], (1)(32](54](9876], (1)(32](654](987], (1][2](9876543], (1](32](987654], (1](432](98765], (1](5432](9876], (21](43](98765], (21](543](9876], (321](654](987]
7 Images	(1)(2)(3)(4)(5)(6)(7)(8][9], (1)(2)(3)(4)(5)(6)(7](98], (1)(2)(3)(4)(5)(6](987], (1)(2)(3)(4)(5)(76](98], (1)(2)(3)(4)(5](9876], (1)(2)(3)(4)(65](987], (1)(2)(3)(4](98765], (1)(2)(3)(54](9876], (1)(2)(3)(654](987], (1)(2)(3](987654], (1)(2)(43](98765], (1)(2)(543](9876], (1)(2](9876543], (1)(32](987654], (1)(432](98765], (1)(5432](9876], (1](98765432], (21](9876543], (321](987654], (4321](98765]
8 Images	(1)(2)(3)(4)(5)(6)(7)(8)(9], (1)(2)(3)(4)(5)(6)(7)(98], (1)(2)(3)(4)(5)(6)(987], (1)(2)(3)(4)(5)(9876], (1)(2)(3)(4)(98765], (1)(2)(3)(987654], (1)(2)(9876543], (1)(98765432], (987654321]
9 Images	(1)(2)(3)(4)(5)(6)(7)(8)(9)

Total Conjugacy classes = 97

RESULTS

From the enumeration above, a summary of the sequence on the number of conjugacy classes of ID_n is listed below:

$2, 4, 7, 12, 19, 30, 45, 67, 97, \ldots$ where n = 1, 2, … A000070 of the Online Encyclopedia of Integer Sequences (OEIS)

Let $a(n)$ be the number of conjugacy classes of ID_n

$a(n)$ =Count 2 for each partition of n and 1 for each decrement. Jon Perry (2004) A000070 of the (OEIS).

For example $a(4)$ = the partitions of 4 are: 4(2), 31(3), 22(2), 211(3), and 1111(2) = 2 + 3 + 2 + 3 + 2 = 12

Also $a(n)$ is the sum of the number of distinct partitions of a positive integer including zero. Sloane, N. J. A. (A000070) of the OEIS. By convention, the number of partition of 0 is 1.

For example, $a(4)$ = no of partitions of 0 + no of partitions of 1 + no of partitions of 2 + no of partitions of 3 + no of partitions of 4 = 1 + 1 + 2 + 3 + 5 = 12.

More generally, the number of Conjugacy classes of ID_n is given as:

$$a(n) = \frac{1}{n}\sum_{k=1}^{n}(S(k)+1)a(n-k) \text{ where } a(0) = 1 \text{ and } S(k) \text{ is the sum of divisors of } k.$$

Vladeta Jovovic (2002), A000070 of the OEIS.
For sum of divisors of n, for example, S(8) = 1 + 2 + 4 + 8 = 15. For higher values of n, we use the formula:

$$S(k) = \prod_{i=1}^{m}\frac{p_i^{q_i+1}-1}{p_i-1} : n = p_1^{q_1}, p_2^{q_2}, \ldots, p_m^{q_m} \text{ where the ps are distinct primes.}$$

(Charles, 2001)

$$a(4) = \frac{1}{4}(2\times 7 + 4\times 4 + 5\times 2 + 8\times 1)$$
$$= \frac{1}{4}(48) = 12$$

CONCLUSION

It has been shown that the number of conjugacy classes in ID_n, for $n \geq 1$ can be calculated using the formula:

$$a(n) = \frac{1}{n}\sum_{k=1}^{n}(S(k)+1)a(n-k) \text{ where } a(0) = 1 \text{ and } S(k) \text{ is the sum of divisors of } k.$$

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