An Order Five Implicit 3-StepBlock Method for Solving Ordinary Differential Equations.

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ABSTRACT

In this paper, self-starting block hybrid method of order $(5,5,5,5)^T$ is proposed for the solution of the special second order ordinary differential equations of the form $y'' = f(x,y),\ a \leq x \leq b$ with associated initial or boundary conditions. The continuous hybrid formulations enable us to differentiate and evaluate at some grids and off – grid points to obtain four discrete schemes, which were used in block form for parallel or sequential solutions of the problems. The computational burden and computer time wastage involved in the usual reduction of second order problem into system of first order equations are avoided by this approach. Furthermore, a stability analysis and efficiency of the block method is presented and the schemes are tested on stiff and non-stiff ordinary differential equations whose solutions are oscillatory or nearly periodic in nature, and the results obtained compared favorably with the exact solution.

(Keywords: block method, hybrid, initial value problem, linear multistep method, self-starting)

INTRODUCTION

Let us consider the numerical solution of the special second order ordinary differential equation of the form:

$$y'' = f(x,y),\quad a \leq x \leq b \qquad \textbf{(1)}$$

with associated initial or boundary conditions. The mathematical models of most physical phenomena, especially in mechanical systems without dissipation, leads to special second order initial value problem of type (1). Solutions to initial value problem of type (1) according to Fatunla [3,4] are often highly oscillatory in nature and thus, severely restrict the mesh size of the conventional linear multistep method. Such system often occurs in mechanical systems without dissipation, satellite tracking, celestial mechanics, Henrici [7].

Lambert [8] and several authors, have written on conventional linear multistep method:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j},\ k \geq 2 \qquad \textbf{(2)}$$

or compactly in the form:

$$\rho(E) y_n = h^2 \delta(E) f_n \qquad \textbf{(3)}$$

where E is the shift operator specified by, $E^j y_n = y_{n+j}$ while ρ and δ are characteristics polynomials and are given as:

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j,\ \delta(\xi) = \sum_{j=0}^{k} \beta_j \xi^j \qquad \textbf{(4)}$$

y_n is the numerical approximation to the theoretical solution y(x) and $f_n = f(x_n, y_n)$.

In the present consideration, our motivations for the study of this approach is a further advancement in efficiency, i.e obtaining the most accuracy per unit of computational effort, that can be secured with the group of methods proposed in this paper over Awoyemi [2], and Yahaya and Mohammed [11]. However, some authors have proposed solutions to special second order initial value problems of ordinary differential equations using different approaches. In particular, Onumanyi etal [9], Awoyemi [2], Yahaya and Adegboye [10], Fudziah *et al*. [6], and Yahaya and Mohammed [11] developed linear multistep methods for special second order ordinary differential equations.

Definition 1: Consistent Lambert [8]

The linear multistep method (2) is said to be consistent if it has order $p \geq 1$, that is if,

$$\sum_{j=0}^{k} \alpha_j = 0 \text{ and } \sum_{j=0}^{k} j\,\alpha_j - \sum_{j=0}^{k} \beta_j = 0 \qquad \textbf{(5)}$$

Introducing the first and second characteristics polynomials (4), we have from (5) LMM type (2) is consistent if $\rho(1) = 0,\ \rho^1(1) = \delta(1)$.

Definition 2: Zero stability Lambert [8]

A linear multistep method type (2) is zero stable provided the roots $\xi_j, j = 0(1)k$ of first characteristics polynomial $\rho(\xi)$ specified as $\rho(\xi) = \det\left|\sum_{j=0}^{k} A(i)\xi^{(k-i)}\right| = 0$ satisfies $|\xi_j| \leq 1$ and for those roots with $|\xi_j| = 1$ the multiciplicity must not exceed two. The principal root of $\rho(\xi)$ is denoted by $\xi_1 = \xi_2 = 1$.

Definition 3: Convergence Lambert [8]

The necessary and sufficient conditions for the linear multistep method type (2) is said to be convergent if it is consistent and zero stable.

Definition 4: Order and Error Constant Lambert [8]

The linear multistep method type (1.4) is said to be of order p if $c_0 = c_1 = \cdots c_{p+1} = 0$ but $c_{p+2} \neq 0$ and c_{p+2} is called the error constant, where $c_0 = \sum_{j=0}^{k} \alpha_j = \alpha_0 + \alpha_1 + \alpha_2 + \ \dots\ + \alpha_k$

$$c_1 = \sum_{j=0}^{k} j\,\alpha_j = (\alpha_1 + 2\,\alpha_2 + 3\,\alpha_3 + \ \dots\ + k\,\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$c_2 = \sum_{j=0}^{k} \frac{1}{2!} j^2\,\alpha_j - \sum_{j=0}^{k} \beta_j = \begin{Bmatrix} \frac{1}{2!}(\alpha_1 + 2^2\,\alpha_2 + 3^3\,\alpha_3 + \ \dots\ + k^2\,\alpha_k) \\ -(\beta_1 + 2\beta_2 + 3\beta_3 + \ \dots + k\beta_k) \end{Bmatrix}$$

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$$c_q = \sum_{j=1}^{k} \{\frac{1}{q!} j^q\,\alpha_j - \frac{1}{(q-2)!} j^{q-2}\beta_j\}$$

$$= \begin{Bmatrix} \frac{1}{q!}(\alpha_1 + 2^q\,\alpha_2 + 3^q\,\alpha_3 + \ \dots\ + k^q\,\alpha_k) \\ -\frac{1}{(q-1)!}(\beta_1 + 2^{(q-1)}\beta_2 + 3^{(q-1)}\beta_3 + \cdots + k^{(q-1)}\beta_k) \end{Bmatrix} \qquad \textbf{(6)}$$

Theorem 1: Lambert [8]

Let $f(x, y)$ be defined and continuous for all points (x, y) in the region D defined by $\{(x, y) : a \leq x \leq b, -\infty < y < \infty\}$ where *a and b* finite, and let there exist a constant L such that for every x, y, y^* such that (x, y) *and* (x, y^*) are both in D:

$$|f(x, y) - f(x, y^*)| \leq L|\,y - y^*| \qquad \textbf{(7)}$$

Then if η is any given number, there exist a unique solution $y(x)$ of the initial value problem (1), where $y(x)$ is continuous and differentiable for all (x, y) *in D*.

The Inequality (7) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

DERIVATION OF THE PROPOSED METHOD

We proposed an approximate solution to (1) in the form:

$$y(x) = \sum_{j=0}^{t+m-1} a_j x^i = y_{n+j}, i = 0(1)m + t - 1 \qquad \textbf{(8)}$$

$$y''(x) = \sum_{j=0}^{t+m-1} i(i-1)a_j x^{i-2} = f_{n+j}, \quad i = 2,3, \dots, m + t - 1 \qquad \textbf{(9)}$$

with m = 4, t = 3 and p = m+t-1

where the $a_j, j = 0,1, (m + t - 1)$ are the parameters to be determined, t and m are points of interpolation and collocation, respectively.

Where P, is the degree of the polynomial interpolant of our choice.

Specifically, we collocate Equation (9) at $\{x_n, x_{n+1}, x_{n+2}, x_{n+\frac{4}{3}}, x_{n+3}\}$ and interpolate Equation (8) at $\{x_n, x_{n+1},\}$ using the method described above; we obtained a continuous form for the solution:

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j}$$

This gives rise to the system of equations put in the matrix form.

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 \\ 0 & 0 & 2 & 6x_{n+\frac{4}{3}} & 12x_{n+\frac{4}{3}}^2 & 20x_{n+\frac{4}{3}}^3 & 30x_{n+\frac{4}{3}}^4 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_{\frac{4}{3}} \\ \beta_3 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+\frac{4}{3}} \\ f_{n+3} \end{bmatrix} \quad \textbf{(10)}$$

The matrix D in Equation (10), which when solved either by matrix inversion techniques or Gaussian elimination method to obtain the values of the parameters $\alpha_j, j = 0(1)m + t - 1$ and then substituting them into Equation (8) give a scheme expressed in the form:

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + h^2[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_{4/3}(x)f_{n+4/3} + \beta_2(x)f_{n+3}] \quad \textbf{(11)}$$

We set $\varphi = (x - x_{n+1})$

$$y(x) = \left\{-\left(\frac{\varphi}{h}\right)\right\} y_n + \left\{\left(\frac{h + \varphi}{h}\right)\right\} y_{n+1}$$

$$+ \left\{\frac{6(\varphi)^6 - 30h(\varphi)^5 + 45h^2(\varphi)^4 - 20h^3(\varphi)^3 + 101h^5(\varphi)}{1440h^4}\right\} f_n$$

$$+ \left\{\frac{-6(\varphi)^6 + 21h(\varphi)^5 + 5h^2(\varphi)^4 - 70h^3(\varphi)^3 + 60h^4(\varphi)^2 + 108h^5\varphi}{120h^4}\right\} f_{n+1}$$

$$+ \left\{\frac{-6(\varphi)^6 + 12h(\varphi)^5 + 25h^2(\varphi)^4 - 20h^3(\varphi)^3 + 27h^5(\varphi)}{240h^4}\right\} f_{n+2}$$

$$+ \left\{\frac{54(\varphi)^6 - 162h(\varphi)^5 - 135h^2(\varphi)^4 + 540h^3(\varphi)^3 - 459h^5(\varphi)}{800h^4}\right\} f_{n+\frac{4}{3}}$$

$$+ \left\{\frac{6(\varphi)^6 - 3h(\varphi)^5 - 15h^2(\varphi)^4 + 10h^3(\varphi)^3 - 16h^5(\varphi)}{1800h^4}\right\} f_{n+3} \quad \textbf{(12)}$$

Evaluating the continuous scheme of Equation (12) at some selected points yield the following schemes:

$$\text{(a) } y_{n+\frac{4}{3}} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{h^2}{437400}\left\{\begin{matrix} 10135f_n + 146580f_{n+1} + 15690f_{n+2} \\ -73953f_{n+4/3} - 1252f_{n+3} \end{matrix}\right\}$$

$$\text{(b) } y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{1200}\left\{\begin{matrix} 85f_n + 1180f_{n+1} + 190f_{n+2} \\ -243f_{n+4/3} - 12f_{n+3} \end{matrix}\right\}$$

$$\text{(c) } y_{n+3} - 3y_{n+1} + 2y_n = \frac{h^2}{1200}\left\{\begin{matrix} 155f_n + 2640f_{n+1} + 1470f_{n+2} - 729f_{n+\frac{4}{3}} \\ +64f_{n+3} \end{matrix}\right\}$$

(13)

Taking the first derivative of Equation (12), thereafter, evaluate the resulting continuous polynomial solution at $x = x_0$ yields:

$$\text{(d) } hz_0 - y_{n+1} + y_n = \frac{h^2}{7200}\left\{\begin{matrix} -1625f_n - 6060f_{n+1} - 1110f_{n+2} + 5103f_{n+\frac{4}{3}} \\ +92f_{n+3} \end{matrix}\right\}$$

(14)

RESULTS AND DISCUSSIONS

Recall, that, it is a desirable property for a numerical integrator to produce solution that behave similar to the theoretical solution to a problem at all times. Thus, several definitions, which call for the method to possess some "adequate" region of absolute stability can be found in several literature. See Lambert [8], Fatunla [3,4] etc.

Order and Error Constant of the Proposed Method

Consider the schemes obtained in Equations (13) and (14), Taylor series expansion about y(x) was used, which constitute the member of a zero stable block integrators of order $(5,5,5,5)^T$ with:

$$c_7 = \left(\frac{2351}{3936600}, \frac{7}{3600}, \frac{1}{600}, -\frac{143}{50400}\right).$$

The application of the block integrators with n = 0 gives the accurate values of unknown as shown in Tables 1 and 2 of this paper.

To start the IVP integration on the sub interval $[X_0, X_4]$, we combine Equation (13) and (14), when n = 0 i.e the 1-block 4-point method are given in Equation (15). Thus produces simultaneously values for y_1, y_2, y_3 and $y_{\frac{4}{3}}$ without recourse to any predictor like Aladeselu [1] and Awoyemi [2] to provide y_1 and y_2 in the main method. Hence this is an improvement over these reported works. Though, this does not becloud the contribution of these authors.

Stability Analysis

Following Fatunla [3,4], the four point block integrator proposed in this report in Equation (13) and (14) are put in the matrix equation form and for easy analysis the result was normalized to obtain:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{4}{3}}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&-\frac{1}{3}&1\\0&0&-1&3\\0&0&-2&6\\0&0&-1&3\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-\frac{5}{3}}\\y_{n-1}\\y_n\end{bmatrix}$$

$$+h^2\left\{\begin{bmatrix}-\frac{913}{5400}&\frac{523}{14580}&-\frac{313}{109350}&-\frac{14}{81}\\-\frac{81}{400}&\frac{19}{120}&-\frac{1}{100}&-1\\-\frac{243}{400}&\frac{49}{40}&\frac{4}{75}&-4\\\frac{567}{800}&-\frac{37}{240}&\frac{23}{1800}&\frac{1}{6}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{4}{3}}\\f_{n+2}\\f_{n+3}\end{bmatrix}+\begin{bmatrix}0&0&0&\frac{2027}{87480}\\0&0&0&\frac{17}{240}\\0&0&0&\frac{31}{240}\\0&0&0&-\frac{65}{288}\end{bmatrix}\begin{bmatrix}f_{n-2}\\f_{n-\frac{5}{3}}\\f_{n-1}\\f_n\end{bmatrix}\right\} \quad (15)$$

with $y_0 = \binom{y_0}{hz_0}$ usually giving along the initial value problem.

Equation (15) is the proposed method. The first characteristics polynomial of the proposed method is given by:

$$\rho(\lambda) = \det[\lambda I - A_1^{(1)}] \quad \textbf{(16)}$$

Solving the determinant of Equation (16), yields $\rho(\lambda) = \lambda^3(\lambda - 1)$, which implies, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ or $\lambda_4 = 1$

By definition of zero stable and Equation (16), the proposed method is zero stable and is also consistent as its order $(5,5,5,5)^T > 1$, thus, it is convergent following Henrici [7] and Fatunla [4].

Numerical Experiments

In what follows, we present some numerical results on some problems.

Problem 1: From Yahaya and Mohammed (2010)

$y'' = -100y$; $y(0) = 1$, $y'(0) = 10$

Table 1: Results for the Proposed Method with one Off- Grid Point at Collocation.

H	X	Exact Value	Approximate Value	Yahaya and Mohammed (2010)	Error of Proposed Method
0	0	1	1	0	0
0.001	0.001	1.009949834	1.009949833	3.00000E-09	1.00000E-09
	0.002	1.019798673	1.019798672	3.00000E-09	1.00000E-09
	0.003	1. 029545534	1. 029545532	4.00000E-09	2.00000E-09
	0.00133	1. 013211164	1.013244050		3. 28860E-05
0.0025	0.0025	1.024684912	1. 024684912	3.00000E-09	0.00000E+00
	0.005	1.048729430	1. 048729430	1.00000E-09	0.00000E+00
	0.0075	1. 072118525	1. 072118524	5.00000E-09	1. 00000E-09
	0.00333	1. 032739452	1. 032771656		3. 22040E-05
0.005	0.005	1.048729430	1. 048729428	1.00000E-09	2.00000E-09
	0.01	1.094837582	1. 094837581	3.00000E-09	1.00000E-09
	0.015	1. 138209210	1. 138209208	1.00000E-09	2. 00000E-09
	0.00667	1. 064426934	1. 064395894		3. 10400E-05

Problem 2: Consider a Non-Linear IVP; $y'' = 2y^3; y(1) = 1, y'(1) = -1$, whose exact solution is $y(x) = 1/x$.

Table 2: Results for the Proposed Method with one Off-Grid Point at Collocation.

N	x	Exact Value	Approximate Value	Awoyemi (1998)	Error of Proposed Method
0	1	1	1	0	0
1	1.1	0.909090109	0.9090914826	2.8483722E-03	1.37360E-06
2	1.2	0.833333333	0.8333348875	2.26883436E-01	1.55450E-06
3	1.3	0.769230769	0.7692330259	7.3968630E+00	2.25690E-06
4	1.4	0.714285714	0.7142880945	2.1168783E-01	2.38050E-06
5	1.5	0.666666667	0.6666693006	3.3156524E-01	2.63360E-06
6	1.6	0.625	0.6250029040	4.3968593E-01	2.90400E-06
7	1.7	0.588235294	0.5882382492	5.3903097E-01	2.95520E-06
8	1.8	0.555555556	0.5555586357	6.3121827E-01	3.07970E-06
9	1.9	0.526315789	0.5263190397	7.1723621E-01	3.25070E-06
10	2.0	0.5	0.5000032814	7.9776590E-01	3.28140E-06

CONCLUSION

Onumanyi et al. [9], and Awoyemi [2] discussed in some detail theoretical and practical aspects of collocation with piecewise polynomial function. Roughly, their results particularly, Awoyemi [2], indicate that the solution of a second order non-linear problem can be approximated with linear multistep methods. In this paper we developed a uniform order $(5,5,5,5)^T$. The resultant numerical integrators possess the following desirable properties:

i. Zero stability i.e. stability at the origin

ii. Convergent schemes

iii. An additional equation from the use of first derivative

iv. Being self – starting as such it eliminate its implementation in predictor – corrector mode

v. Facility to generate solutions at 4 points simultaneously

vi. Produce solution over sub intervals that do not overlaps.

In addition, the new schemes compares favorably with the theoretical solution and the results are more accurate than Yahaya and Mohammed [11]; and Awoyemi [2] (see Table 1 and 2).

Hence, our work is an improvement over other cited works.

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SUGGESTED CITATION

Yahaya, Y.A. and A.M. Sagir. 2013. "An Order Five Implicit 3-Step Block Method for Solving Ordinary Differential Equations". *Pacific Journal of Science and Technology*. 14(1):176-181.

Pacific Journal of Science and Technology