

The RAS of Ibijola and Sunday (2010) is a Sub-Region of the RAS of Odekunle and Sunday (2012).

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ABSTRACT

One numerical integrator is more stable than the other, if the latter has a larger absolute stability region than the former. In this research paper, we show that the Region of Absolute Stability (RAS) of Ibijola and Sunday (2010) is a sub region of the RAS of Odekunle and Sunday (2012). Both numerical integrators were constructed by locally representing the theoretical solution $y(x)$ to first-order initial value problems in the interval $[x_n, x_{n+1}]$ by a nonlinear interpolating function.

We further show that the Interval of Absolute Stability (IAS) of Ibijola and Sunday (2010) is a sub region of that of Odekunle and Sunday (2012).

(Keywords: Interval of Absolute Stability, IAS, Region of Absolute Stability, RAS, Region of Instability, RIS, stability function, sub-region)

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INTRODUCTION

To determine whether a numerical integrator will produce reasonable results or not, we need a notion of stability that is different from zero-stability. There are a wide variety of other forms of stability that have been studied in various contexts. The one that is most basic is that of absolute stability (Leveque, 2007).

Odekunle and Sunday (2012) derived a new numerical integrator (as an improvement over that of Ibijola and Sunday 2010) capable of solving problems of the forms:

$$y' = f(x, y), y(x_0) = y_0, x > 0 \quad (1a)$$

and their equivalent systems,

$$\left. \begin{array}{l} y_1' = f_1(x, y_1, y_2, \dots, y_m), y_1(a) = \eta_1 \\ y_2' = f_2(x, y_1, y_2, \dots, y_m), y_2(a) = \eta_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m' = f_m(x, y_1, y_2, \dots, y_m), y_m(a) = \eta_m \end{array} \right\}, x > 0 \quad (1b)$$

The numerical integrator is developed based on the assumption that the theoretical solution $y(x)$ to equation (1a) can be locally represented in the interval $[x_n, x_{n+1}]$ by a nonlinear interpolating function comprising of the combination of power series, exponential function and cyclometric function, given by:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3e^{\alpha x} + b \sin x \quad (2)$$

where $a_0, a_1, a_2, a_3, b,$ and α are undetermined coefficients. The integrator is given by,

$$y_{n+1} = y_n + h \left[\begin{aligned} & \left[f_n - \left[f_n' - (f_n^2 + f_n^3 \cot(x_n)) \left(\frac{1}{\alpha} + x_n \right) + f_n^3 \right] x_n \right. \\ & \left. - (f_n^2 + f_n^3 \cot(x_n)) \left(\frac{1}{\alpha^2} + \frac{x_n}{\alpha} + \frac{x_n^2}{2} \right) - f_n^3 \cot(x_n) \right] \\ & + \left[f_n' - (f_n^2 + f_n^3 \cot(x_n)) \left(\frac{1}{\alpha} + x_n \right) + f_n^3 \right] \left(x_n + \frac{h}{2} \right) \\ & + (f_n^2 + f_n^3 \cot(x_n)) e^{\alpha x_n} \left(\frac{1}{\alpha^2} + \frac{h}{(2!)(\alpha)} + \frac{h^2}{3!} \right) \\ & + f_n^3 \left[-\frac{h}{2!} + \frac{h^3}{4!} - \frac{h^5}{6!} + \cot(x_n) \left(1 - \frac{h^2}{3!} + \frac{h^4}{5!} - \frac{h^6}{7!} \right) \right] \end{aligned} \right] \quad (3a)$$

This is a one-step numerical integrator capable of solving problems of the form (1a & 1b). For the implementation of this scheme, see Sunday and Odekunle (2012). This type of construction was first reported in Fatunla (1976). He proposed a numerical integrator which is particularly well suited to solve problems of the form (1a) having oscillatory or exponential solutions.

This method was based on the local representation of the theoretical solution $y(x)$ to the IVP (1a) in the interval $[x_n, x_{n+1}]$ by a nonlinear polynomial interpolating function $y(x) = a_0 + a_1x + breale^{(\rho x + \mu)}$, where a_0, a_1 and b are real undetermined coefficients, while ρ and μ are complex parameters.

Ibijola (1997) in his Doctoral dissertation under the supervision of Fatunla extended the work of Fatunla by proposing a numerical integration scheme suited for IVPs of the form (1a) in the interval $[x_n, x_{n+1}]$ by a nonlinear polynomial interpolating function $y(x) = a_0 + a_1x + a_2x^2 + b(reale^{(\rho x + \mu)})$, where a_0, a_1, a_2 , and b are real undetermined coefficients and ρ and μ are complex parameters.

In 2010, Ogunrinde in her Doctoral dissertation extended the work of Ibijola (1997), by representing the theoretical solution $y(x)$ to the

IVP (1a) in the interval $[x_n, x_{n+1}]$ by a nonlinear polynomial interpolating function $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + breale^{(\rho x + \mu)}$, where a_0, a_1, a_2, a_3 and b are real undetermined coefficients, while ρ and μ are complex parameters.

The numerical results were simply remarkable. In 2010 also, Ibijola and Sunday (2010) derived a numerical integrator by representing the theoretical solution $y(x)$ to the IVP (1a) in the interval $[x_n, x_{n+1}]$ by a nonlinear polynomial interpolating function $y(x) = a_0 + a_1e^{\alpha x}$, where a_0, a_1 and α are real undetermined coefficients. The integrator is given by:

$$y_{n+1} = y_n + \frac{f_n}{\alpha} (e^{\alpha h} - 1) \quad (3b)$$

Other integrators developed based on nonlinear Interpolants include those of Ibijola and Imoni (2004), Ibijola, Bamisile and Sunday (2011), among others.

The absolute stability properties of (3a and 3b) can be investigated by applying (1a) to the well-known A-stability model equation, otherwise known as the scalar test equation:

$$y' = \lambda y, y(x_0) = y_0 \quad (4)$$

where λ is a complex constant with negative real part, $\text{Re}(\lambda) < 0$ and that the equation has a steady state solution:

$$y(x) = y(x_0)e^{\lambda x} \quad (5)$$

Thus, a desired requirement of any integrator would be that the steady solution $y(x)$ of the test equation agrees, as much as possible, with the solution of the associated difference equation that approximates the differential equation (1a). On the application of (1a) to the scalar test problem (4), we obtain the first order difference equation,

$$y_{n+1} = \mu(z)y_n, z = \lambda h \quad (6)$$

where the stability function $\mu(z)$ is either a polynomial in z (for an explicit method) or a rational function in z (for an implicit method). The parameter in (3a) may be chosen to ensure that $\mu(z)$ is an approximation to e^z near $z = 0$, and if it is p th order accurate, then:

$$\mu(z) - e^z = O(z^{p+1}), \text{ as } z \rightarrow 0 \quad (7)$$

Definition 1 (Fatunla, 1988)

The one-step scheme (3a and 3b) is said to be absolutely stable at a point z in the complex plane provided the stability polynomial (function) $\mu(z)$ in (6) fulfils the following condition:

$$|\mu(z)| < 1 \quad (8)$$

Definition 2 (Lambert, 1973)

A region D of the complex plane is said to be a Region of Absolute Stability (RAS) of a given method, if the method is absolutely stable for every $z \in D$, i.e.,

$$RAS = \{z : |\mu(z)| = |\mu(u + iv)| < 1, i = \sqrt{-1}\} \quad (9)$$

We also define the RAS as a region in the complex z plane, where $z = \lambda h$. It is those values of z such that the numerical solutions of

$y' = \lambda y$ satisfy $y_n \rightarrow 0$ as $n \rightarrow \infty$ for any initial condition.

Definition 3 (Fatunla, 1988)

The numerical integration schemes (3a and 3b) is said to be A-stable if its RAS, defined in (9) contains the entire left-half of the complex z - plane i.e. $\text{Re}(z) < 0$ or $\text{Re}(\lambda h) < 0$.

A-stability is a very desirable property for any numerical integration algorithm, particularly if the IVP (1a) were to be stiff or stiff oscillatory.

Definition 4 (Fatunla, 1976)

A given one-step method is said to be L-stable if it is A-stable and in addition when applied to the scalar test equation (4), it yields (6) where $|\text{Re}(\lambda h)| \rightarrow 0$.

THE STABILITY REGIONS

Using the facts that,

$$\left. \begin{aligned} f_n &= \lambda y_n \\ f_n' &= \lambda f_n = \lambda^2 y_n \\ f_n^2 &= \lambda f_n' = \lambda^3 y_n \\ f_n^3 &= \lambda f_n^2 = \lambda^4 y_n \end{aligned} \right\} \quad (10)$$

$$e^{\alpha x_n} = 1 + \alpha x_n + \frac{\alpha x_n^2}{2!} + \frac{\alpha x_n^3}{3!} + \dots \quad (11)$$

and

$$\cot(x_n) = x_n^{-1} - \frac{x_n}{3} - \frac{x_n^3}{45} - \frac{2x_n^5}{945} - \dots \quad (12)$$

Equation (3a) reduces to,

$$y_{n+1} = y_n + \lambda h y_n + \frac{h^2 \lambda^2 y_n}{2} + \frac{h^3 \lambda^3 y_n}{6} + \frac{h^4 \lambda^4 y_n}{24} \quad (13)$$

$$y_{n+1} = y_n \left(1 + \lambda h + \frac{h^2 \lambda^2}{2} + \frac{h^3 \lambda^3}{6} + \frac{h^4 \lambda^4}{24} \right) \quad (14)$$

If we put $\lambda h = z$, we obtain,

$$\mu(z) = \frac{y_{n+1}}{y_n} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} \quad (15)$$

Equation (15) is the stability polynomial of Odekunle and Sunday (2012).

Theorem

The RAS of Ibijola and Sunday (2010) is a sub-region of the RAS of Odekunle and Sunday (2012).

Proof

The stability function of Ibijola and Sunday (2010) is:

$$\mu(z) = 1 + z \quad (16)$$

Substituting $z = u + iv$ in equation (16), we have:

$$\mu(u, v) = 1 + u + iv \quad (17)$$

Yielding,

$$\mu(u, v) = \left(\frac{24 + 24u + 12u^2 - 12v^2 + 4u^3 - 12uv^2 + u^4 - 6u^2v^2 + v^4}{24} \right) + (i) \left(\frac{24v + 24uv + 12u^2v - 4v^3 + 4u^3v - 4uv^3}{24} \right) \quad (23)$$

on the substitution of $z = u + iv$ in (22). The use of polar transformations in (18) leads us to equation (19), if and only if,

$$R \left[\begin{array}{l} R^7 + 8R^6 \cos \theta + R^5 (40 \cos^6 \theta + 144 \cos^4 \theta \sin^2 \theta + 24 \cos^2 \theta \sin^4 \theta) \\ -8 \sin^6 \theta + R^4 (144 \cos^5 \theta - 96 \cos^3 \theta \sin^2 \theta - 48 \cos \theta \sin^4 \theta) \\ + 384R^3 \cos^4 \theta + 768R^2 \cos^3 \theta + 1152R \cos^2 \theta + 1152 \cos \theta \end{array} \right] \leq 0 \quad (24)$$

By employing the transformations:

$$u = R \cos \theta \text{ and } v = R \sin \theta \quad (18)$$

we obtain,

$$|\mu(R, \theta)| \leq 1 \quad (19)$$

if and only if,

$$R[2 \cos \theta + R] \leq 0 \quad (20)$$

Define,

$$F(R, \theta) = R[2 \cos \theta + R] \quad (21)$$

Then, $F(R, \theta) = 0$ is the Jordan curve of the RAS. In this case, $F(R, \theta) < 0$ representing the interior of the Jordan curve is the RAS of Ibijola and Sunday (2010) while $F(R, \theta) > 0$ representing the exterior of the Jordan curve is the Region of Instability (RIS) of Ibijola and Sunday (2010).

For Odekunle and Sunday (2012), the stability function is,

$$\mu(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} \quad (22)$$

Our Jordan curve $F(R, \theta)$ in this case is given by Equation (24). Hence, $F(R, \theta) < 0$ representing the interior of the Jordan curve is the RAS of Odekunle and Sunday (2012) while $F(R, \theta) > 0$ representing the exterior of the Jordan curve is the RIS of Odekunle and Sunday (2012).

The curves of the two functions are both located in the second and third quadrants of the U-V plane. They both pass through the origin. For Ibijola and Sunday (2010), the curve has its radial values $+0$ at 90° and -0 at 270° . The horizontal diameter is the straight line with end points $(-2, 0)$ and $(0, 0)$. The polar form $|\mu(R, \theta)| \leq 1 \Leftrightarrow R[R + 2\cos\theta] \leq 0$ gives the interior of the unit circle centered at $(-1, 0)$.

Therefore, the IAS of Ibijola and Sunday is given by $(-2, 0)$.

On the other hand, Odekunle and Sunday (2012) has its radial values $2\sqrt{2}$ at 90° and $-2\sqrt{2}$ at 270° ; rising to its absolute peak variationally at 180° . Let,

$R(\text{Ibijola and Sunday 2010})$
and $R(\text{Odekunle and Sunday 2012})$

respectively denote the radial values of the Jordan curve of Ibijola and Sunday (2010) and Odekunle and Sunday (2012) at each point $\theta \in [0, 360^\circ]$, then we have:

$$|R(\text{Ibijola and Sunday 2010})| \leq |R(\text{Odekunle and Sunday 2012})| \quad (25)$$

Hence,

$$\begin{aligned} & \{(R, \theta) : R[2\cos\theta + R], 0 \leq \theta \leq 360^\circ\} \\ \subsetneq & \left\{ (R, \theta) : R \begin{bmatrix} R^7 + 8R^6 \cos\theta + R^5(40\cos^6\theta + 144\cos^4\theta \sin^2\theta + 24\cos^2\theta \sin^4\theta - 8\sin^6\theta) + R^4(144\cos^5\theta \\ -96\cos^3\theta \sin^2\theta - 48\cos\theta \sin^4\theta) + 384R^3 \cos^4\theta + 768R^2 \cos^3\theta + 1152R \cos^2\theta + 1152 \cos\theta \end{bmatrix}, 0 \leq \theta \leq 360^\circ \right\} \end{aligned} \quad (26)$$

That is,

$$RAS \text{ Ibijola and Sunday}(2010) \subsetneq RAS \text{ Odekunle and Sunday}(2012) \quad (27)$$

This completes the proof.

Substituting $\theta = 0^\circ$ in equation (21) for Ibijola and Sunday (2010) we have:

$$F(R, 0^\circ) = R[2 + R] = 0 \quad (28)$$

$$\Rightarrow R = 0 \quad (29)$$

$$\Rightarrow R = -2 \quad (30)$$

Similarly, for Odekunle and Sunday (2012), at $\theta = 0^\circ$ from equation (24), we have:

$$F(R, 0^\circ) = R \begin{bmatrix} 1152 + 1152R + 768R^2 \\ +384R^3 + 144R^4 \\ +40R^5 + 8R^6 + R^7 \end{bmatrix} = 0 \quad (31)$$

$$\Rightarrow R = 0 \quad (32)$$

$$\begin{aligned} & R^7 + 8R^6 + 40R^5 \\ \Rightarrow & +144R^4 + 384R^3 \end{aligned} \quad (33)$$

$$+768R^2 + 1152R + 1152 = 0$$

$$F(-2,0) = (-2) \begin{bmatrix} 1152 + 1152(-2) \\ +768(-2)^2 + 384(-2)^3 \\ +144(-2)^4 + 40(-2)^5 \\ +8(-2)^6 + (-2)^7 \end{bmatrix} \\ = (-2)(256) < 0 \quad (34)$$

So $(-2,0)$ is an interior point of $F(-R,\theta)$,

$$F(-3,0) = (-3) \begin{bmatrix} 1152 + 1152(-3) \\ +768(-3)^2 + 384(-3)^3 \\ +144(-3)^4 + 40(-3)^5 \\ +8(-3)^6 + (-3)^7 \end{bmatrix} \\ = (-3)(-171) > 0 \quad (35)$$

Therefore, $(-3,0)$ is an exterior point of $F(-R,\theta)$. It is important to note that the RAS is the interior of the Jordan curve $F(-R,\theta) = 0$ designated by $F(-R,\theta) < 0$ and the Region of Instability (RIS) is given by $F(-R,\theta) > 0$. Therefore, for Odekunle and Sunday (2012) there exists a point $\xi \in (-3,-2)$ such that $F(\xi,0) = 0$. Thus, the IAS for Odekunle and Sunday (2012) is $IAS = (\xi, 0)$.

We shall now use Matlab to determine the real root ξ that lie between -3 and -2 for Odekunle and Sunday (2012). Denote the polynomial (35) of Odekunle and Sunday (2012) as $P(OS)$ and the roots as $roots_P(OS)$.

```
>> P(OS)=[1 8 40 144 384 768 1152 1152]
P(OS) =
    Columns 1 through 5
         1         8        40       144       384
    Columns 6 through 8
       768       1152       1152
```

```
>> roots_P(OS)=roots (P(OS)) % find the roots of
P(OS)
roots_P(OS) =
   -2.7853
  -2.2194 + 1.6873i
  -2.2194 - 1.6873i
   0.2194 + 2.4753i
   0.2194 - 2.4753i
  -0.6074 + 2.8719i
  -0.6074 - 2.8719i
```

Therefore, the IAS for Ibijola and Sunday (2010) is $IAS = (-2,0)$ and that of Odekunle and Sunday (2012) is $IAS = (-2.7853,0)$. Below, we present the stability functions and IAS of some existing methods (based on non-linear interpolating functions), that of Ibijola and Sunday (2010) and Odekunle and Sunday (2012).

Table 1: Stability Functions and IAS of the Two Integrators and Some other Integrators.

S/No.	Methods	Stability Functions $\mu(z)$	Intervals of Absolute Stability (IAS)
1.	Fatunla (1976), Ibijola and Sunday (2010)	$1+z$	$(-2,0)$
2.	Ibijola (1997)	$1+z+\frac{z^2}{2}$	$(-2,0)$
3.	Ibijola and Ogunrinde (2010)	$1+z+\frac{z^2}{2}+\frac{z^3}{6}$	$(-2.5127, 0)$
4.	Odekunle and Sunday (2012)	$1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}$	$(-2.7853, 0)$

Figures 1 and 2 below respectively show the RAS of Ibijola & Sunday (2010) and Odekunle & Sunday (2012).

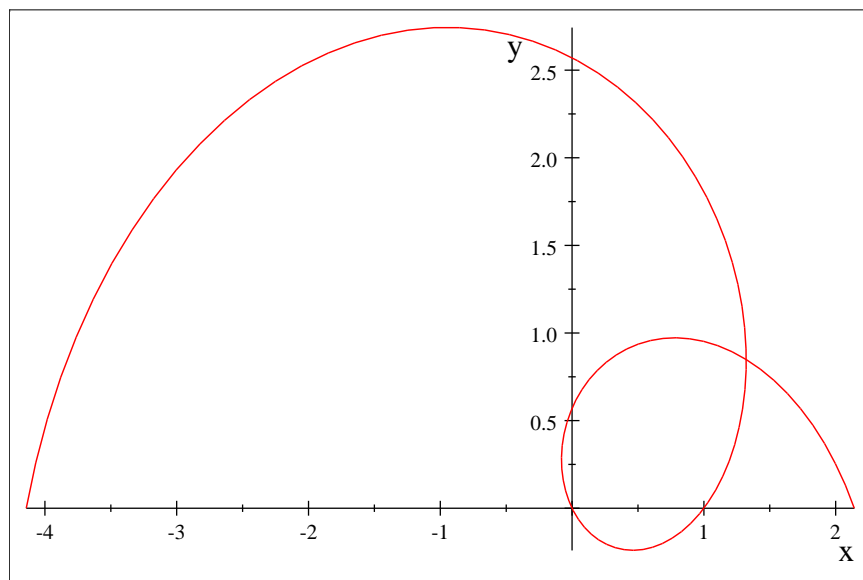


Figure 1: RAS of Ibijola and Sunday (2010).

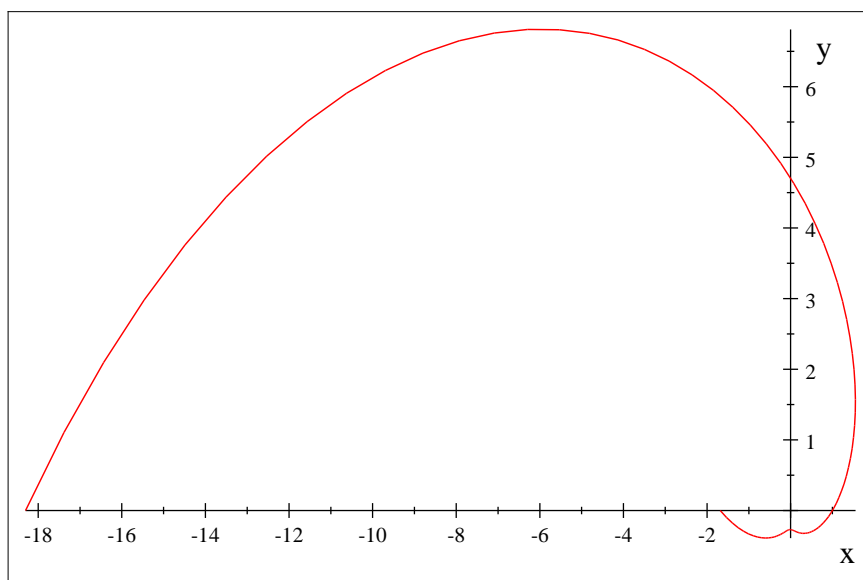


Figure 2: RAS of Odekunle and Sunday (2012).

CONCLUSION

It is obvious from the presentation above that the stability functions and IAS of Ibijola and Sunday (2010) are sub-regions of the stability functions and IAS of Odekunle and Sunday (2012). It is therefore obvious that the new numerical integrator derived by Odekunle and Sunday (2012) will perform better than that of Ibijola and Sunday (2010).

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