The Application of Vectors (in Mathematics) to Determine an Optimal Trajectory for Missiles in Order to Rendezvous with Flying Targets.

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ABSTRACT

The 2- and 3-dimensional co-ordinate geometry systems are discussed. A vectorial concept based on the 2-dimensionhal co-ordinate geometry system is presented. Application of vectors in expressing both the course and velocity of flying objects is presented. Since an enemy aircraft, which is equally a flying object, could be described both in bearing (course) and velocity by vectors, the concept of vectors in determining and defining the bearing of an anti-aircraft missile, to hit such target, is equally presented in this paper through a practical illustration.

(Keywords: vectors, bearing, firing, missiles, aircraft, flying, rendezvous)

INTRODUCTION

In warfare, it is often difficult to fire a conventional missile to hit a flying aircraft precisely at mid-air by simple sighting of the aircraft (Washington Post, 10/02/10). The aircraft, which is the target, most often, could be flying at specific course and altitude but the missile fired at no specific course. If the missile so-fired, hits the target, it is simply by chance, not by pre-determined firing angle (which implies course). What this implies is that since the course, speed, and altitude of the target are mostly known, or could be established, it is hence equally necessary to know or establish the optimum firing angle (course) at which the missile could be fired in order for it to rendezvous with (hit) the target, thereby destroying it.

THE USE OF VECTORS (IN MATHEMATICS) TO DETERMINE POSITION AND BEARING

Any point whether on land, air or sea, can be described vectorially using 2-dimensional or 3-dimensional co-ordinate geometry. In most cases, aircrafts, whether commercial or military, fly by geographic compass which describe its position and bearing (North, South, East, and West), relative to each other. This concept of position and direction determination is similar to 2-dimensional co-ordinate geometry system. The concept of 2-dimensional co-ordinate geometry will therefore be explored for this application.

The 2-Dimensional Co-ordinate Geometry

The 2-dimensional co-ordinate geometry is based on orthogonal concepts which divide a plane of 360 degrees into four (4) equal portions (Smyrl, 2009). The upward vertical direction, from the origin, is conventionally assigned a notation of “j” or “y” while the downward vertical direction, from the origin, is equally conventionally assigned “-j” or “-y”. Similarly, the horizontal direction from the origin to the right, is conventionally assigned a notation of “i” or “x”, while to the left from the origin, it is assigned a notation of “-i" or “-x”. These are all illustrated in Figure 1 below, using the “i” and “j” notation.

As earlier mentioned, most flying objects make use of this co-ordinate system to define their path(s). To hit such targets mid-air, a fired conventional missile should equally make use of this co-ordinate system, to define its path in order to hit the intended target successfully. This application will be illustrated with a practical problem.

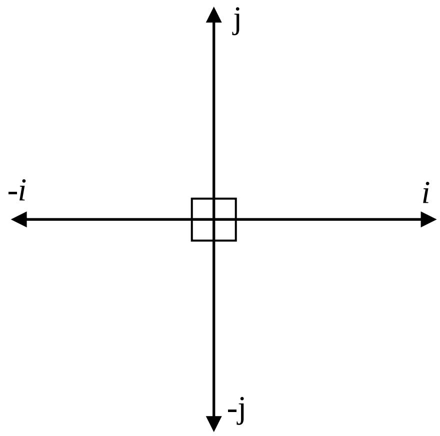

Figure 1: A 2-Dimensional Co-ordinate System.

<u>An Illustration of Determining a Firing Angle (Co-ordinates) of a Missile to Rendezvous with (hit) a Flying Object</u>

A fighter aircraft takes off from point A at 0900 hrs and flies on a steady South-East course at a speed of 15 km/hr at an altitude of 20,000 meters. An anti-aircraft missile stationed at position B, 10 km South of A, is to be fired to hit and destroy the fighter aircraft. Assuming the aircraft does not change its speed and altitude:

(i) If the limiting speed of the missile is 12 km/hr, at what angle should it be fired from point B, at 0900 hrs, in order for it to rendezvous with (hit) the fighter aircraft? At what time will the missile hit the aircraft successfully?

(ii) If for some reason, the missile was not fired (still from point B) at 0900 hrs but fired at 0940 hrs (still from point B), at what angle should it be fired, still at its limiting speed of 12 km/hr, to get as near as possible to the aircraft? What will be this closest distance, and at what time shall it occur?

<u>Solution</u>

(i) The position and speed of the aircraft and the missile have to be described in the i – j, 2-dimensional co-ordinate system as follows:

For the aircraft:

Position, $r_1 = (0)\,i + 10j$ **(1)**

Speed, $V_1 = \frac{15}{\sqrt{2}}\,i - \frac{15}{\sqrt{2}}\,j$ **(2)**

Hence, we have the following space diagram:

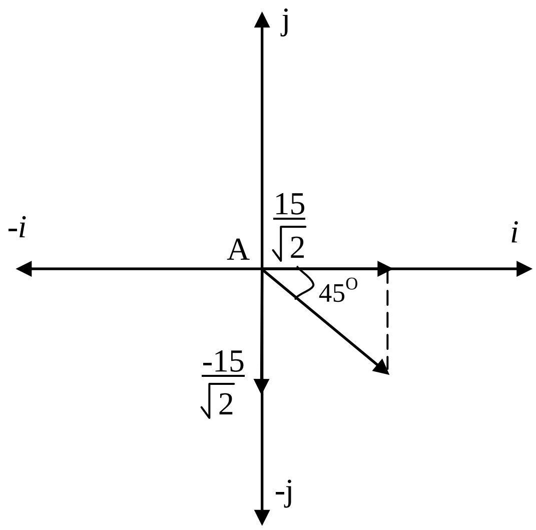

Figure 2: Speed Vector Diagram of the Aircraft for the Missile.

Position, $r_2 = (0)\,i + (0)j$ **(3)**

Since the co-ordinates of the speed of the missile is not known, it is arbitrarily assigned the co-ordinates $(x)\,i + (y)j$.

:. Speed of missile, $V_2 = (x)\,i + (y)j$ **(4)**

from which, $x^2 + y^2 = (12)^2 = 144$ **(5)**

Now, for the missile to hit the aircraft, it (the aircraft) must be given equal but opposite velocity (Craggs, 2008). In other words, it must be theoretically "reduced to rest" at A, and the missile moves with a compounded velocity V, where:

$V = \left(x - \frac{15}{\sqrt{2}}\right) i + \left(y + \frac{15}{\sqrt{2}}\,j\right)$ **(6)**

The following vectorial diagram hence results:

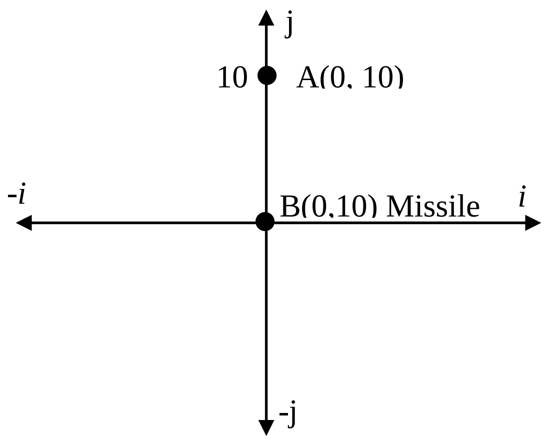

Figure 3: A Vectorial Diagram Reducing the Aircraft to Rest.

Now, to rendezvous with the aircraft “stationary” at A, the missile must move with a speed, V along BA, as shown in Figure 3 above.

From Figure 3:

$$\left(x - \frac{15}{\sqrt{2}}\right) = 0 \text{ (the component of } i \text{ in V is zero)} \quad \textbf{(7)}$$

$$x = \frac{15}{\sqrt{2}}^{\;2} \text{ and}$$

$$x^2 = \left(\frac{15}{\sqrt{2}}\right)^2 = \frac{225}{2} \quad \textbf{(8)}$$

Substituting for x^2 in eqn. (5), we have:

$$y^2 = 144 - \frac{225}{2} = \frac{288 - 225}{2} = \frac{63}{2}$$

$$\therefore\ y = \sqrt{\frac{63}{2}} \quad \textbf{(9)}$$

Hence, the speed and course of the missile is described by:

$$V_2 = \left(\frac{15}{\sqrt{2}}\right) i + \left(\frac{63}{\sqrt{2}}\right) j \quad \textbf{(10)}$$

The following vectorial diagram hence results:

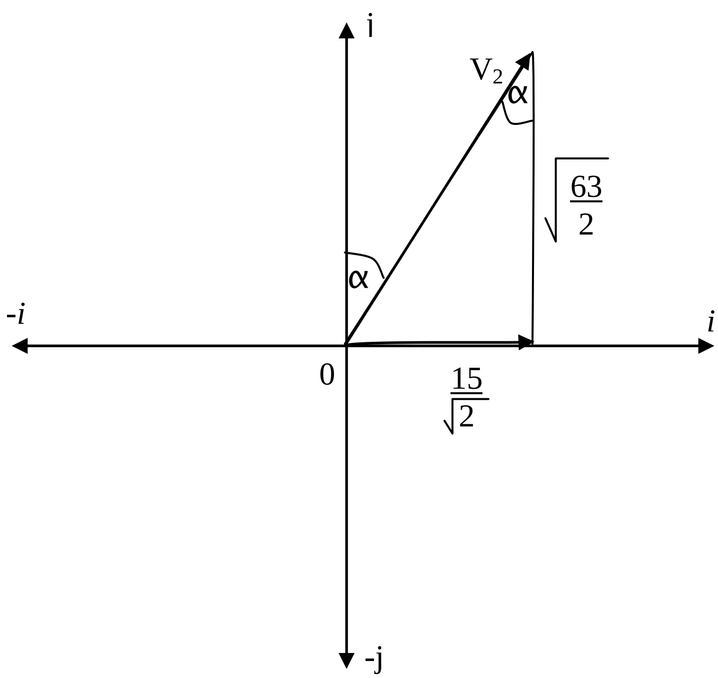

Figure 4: A Vectorial Diagram Illustrating the Speed and Course of the Missile.

From Figure 4:

$$\tan\alpha = \frac{15/\sqrt{2}}{\sqrt{63}/\sqrt{2}} = \frac{15}{\sqrt{63}} = \frac{15}{\left(\sqrt{9}\right)\left(\sqrt{7}\right)} = \frac{15}{3\sqrt{7}}$$

$$\tan\alpha = \frac{5}{\sqrt{7}} = 1.8898 \approx 1.890$$

$$\alpha = \tan^{-1}(1.890) = 62^{o}\,7' \qquad \textbf{(11)}$$

Hence, the course (bearing) at which the missile must be fired in order to hit the aircraft successfully is N62^{o} 7' E. The time of this rendezvous will be:

$$t = \frac{AB}{} = \frac{AB}{V_2} = \frac{AB}{\frac{\sqrt{63}}{\sqrt{2}} + \frac{15}{\sqrt{2}}} = \frac{AB\sqrt{2}}{\left(\frac{3\sqrt{7}}{\sqrt{2}} + \frac{15}{\sqrt{2}}\right)} 3\sqrt{7} + 15$$

:. Time of rendezvous = 0937 hrs **(12)**

Comments on the First Part of the Solution:

It can be seen that the optimum bearing of the missile (the firing angle, Equation (11)), was obtained from the knowledge of the speed and bearing of the aircraft and the speed of the missile. Hence generally, in order to determine the firing angle of the missile, once its limiting speed and that of the aircraft are known, the mathematical procedure above can be coded in any programming language of choice into a computer. This would eventually lead to a numerical value for the firing angle (its bearing).

If the missile was not fired immediately but 40 mins. later, in 40 mins., the aircraft would have covered the following distance:

$$= \left(\frac{15}{\sqrt{2}}\right) i - \left(\frac{15}{\sqrt{2}}\right) j \; \frac{40}{60}$$

$$= \left[\frac{2}{3}\left(\frac{15}{\sqrt{2}}\, i - \frac{15}{\sqrt{2}}\, j\right)\right] \qquad \textbf{(13)}$$

$$= \frac{2}{3}\left[\left(\frac{15}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\right)\right] i - \frac{2}{3}\left[\left(\frac{15}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\right) j\right]$$

Distance $= (5/\sqrt{2})\, i - (5\sqrt{2})\, j$ **(14)**

Its position vector, A_1, will be:

Position vector, A_1 $= (5\sqrt{2})\, i + (10 - 5\sqrt{2})\, j$

$= 7i + 3j$

With this information, we now construct a triangle, ABA_1, as shown in Figure 5.

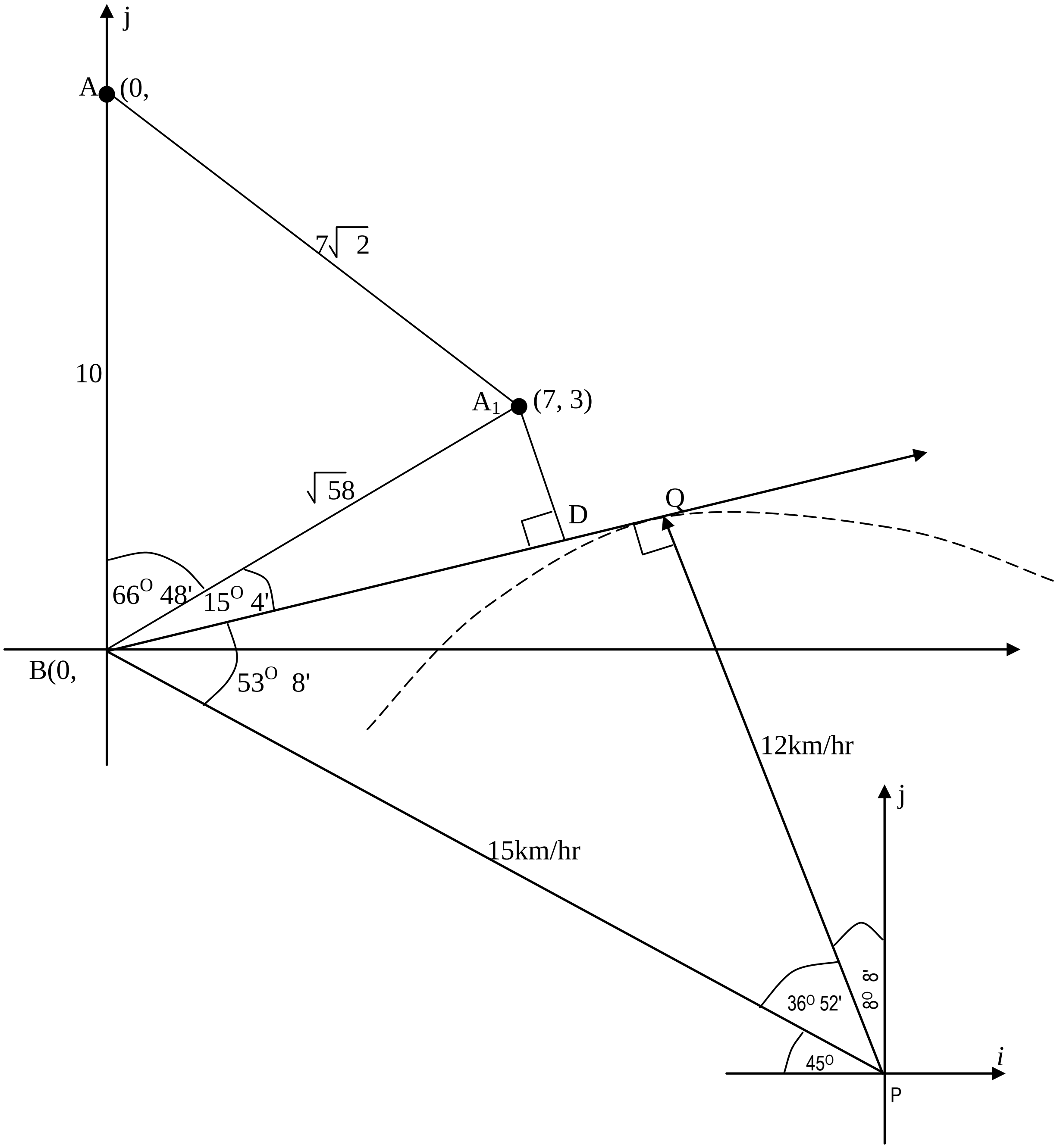

Figure 15: A Composite Vector Diagram showing Position Vector, A_1 of the Aircraft Relative to P, where the Missile is Fired From.

In triangle ABA_1, $AB = 10$, $AA_1 = 7\sqrt{2}$, $BA_1 = \sqrt{58}$

$$\cos \angle ABA_1 = \frac{(10)^2 + (\sqrt{58})^2 - (7\sqrt{2})^2}{2(10)(\sqrt{58})}$$

$$\cos \angle ABA_1 = \frac{100 + 58 - 98}{2(10)(\sqrt{58})} = \frac{60}{20\sqrt{58}} = \frac{3}{\sqrt{58}}$$

$$\Rightarrow \angle ABA_1 = \text{Cos}^{-1} \frac{3}{(\sqrt{58})} = 66^o\ 48'$$

Now, a composite vector diagram for the entire scenario has to be developed. (It has to be compounded with the vectorial diagram of the aircraft, 40 mins. after take-off). This has already been done, and it illustrated in Figure 5. In Figure 5, BPQ is a triangle of velocities such that BQ is perpendicular to PQ.

PQ = Actual velocity of the missile = 12km/hr

PB = Relative velocity of the missile (as it appears to the aircraft).

BQ = Resultant velocity of the missile.

To determine PB and BQ, we proceed as follows:

$PQ\ (= 12) = PB\text{Cos}36^o\ 51'$

$$\therefore \quad PB = \frac{12}{\text{Cos}36^o\ 51'} = 15\text{km/hr}$$

$BQ = 15\text{Sin}\ 36^o\ 52' = 9\text{km/hr}.$

Let the perpendicular from A_1 meet BQ at D, then $\angle A_1BD = 90^o - 66^o\ 48' - 8^o\ 8' = 15^o\ 4'$

$$A_1D = A_1B\text{Sin}(15^o\ 4') = 58\ \text{Sin}\ (15^o\ 4') = 2\text{km}$$

$$A_1D = 2\text{km}$$

Time of closest approach of the missile to the aircraft, t, is:

$$t = \frac{BD}{9} = \frac{\sqrt{58}\ (\text{Cos}15^o\ 4')}{9} = \frac{7.3416}{9} = 0.8173\ \text{hrs}$$

$$t = (0.8173)(60) = 49\ \text{mins.}$$

Hence, the course that the missile has to be fired, 40 mins after the aircraft had taken off, in order for it to get, as near as possible, to the aircraft is:

Course (Bearing) = $N8^o\ 8'\ W$

The closest distance of approach in this respect is 2km, and the time, t_f, that this would occur is:

$$t_f = 9\ \text{hrs}\ \ 40\ \text{mins} + 49\ \text{mins}$$
$$t_f = 10\ \text{hrs}\ 29\ \text{mins} = 10.29\ \text{hrs}$$

This is 42 mins late relative to when the aircraft would have been successfully hit as shown in part 1 of the solution, (Equation (10)).

<u>Comments on the Second Part of the Solution</u>

It would have been noticed that a delay of 40 mins in firing the missile had caused it (the missile) to miss the target (the aircraft) by 2km. That was the nearest it could get to it due to the 40 mins delay.

DISCUSSION

What this illustrative problem demonstrates is that immediately an enemy aircraft is sighted, its velocity and bearing should be immediately and automatically determined using remote sensing devices. Thereafter, by the comments presented under part 1 of the solution (coding the information and procedure into a computer), an optimum firing angle (bearing) of a missile could equally be immediately and automatically determined. Since the optimum velocity of the missile would have been known, the missile could then be "released" (fired automatically) towards its target, the aircraft. This is an applied minimum–time problem (Barnett, 2010).

CONCLUSION

In the paper, vectors (in Mathematics) have been applied to determine an optimum bearing to fire a missile to hit an aircraft. The illustration is actually for defensive purposes, not for offensive purposes. Everyone on earth should join hands to work together to make the earth a peaceful planet to live on. However, since some humans and nations do choose to be offensive against peaceful humans and nations, such humans and nations have the right to defend themselves. In such situations, the strategy contained in this paper could be employed for self-defense.

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