

# Euler's Method for Solving Initial Value Problems in Ordinary Differential Equations.

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## ABSTRACT

This work presents Euler's method for solving initial value problems in ordinary differential equations. This method is presented from the point of view of Taylor's algorithm which considerably simplifies the rigorous analysis. We discuss the stability and convergence of the method under consideration and the result obtained is compared to the exact solution. The error incurred is undertaken to determine the accuracy and consistency of Euler's method.

(Keywords: differential equation, Euler's method, error, convergence, stability)

## INTRODUCTION

Differential equations can describe nearly all systems undergone change. They are ubiquitous in science and engineering as well as economics, social science, biology, business, etc. Many mathematicians have studied the nature of these equations and many complicated systems can be described quite precisely with compact mathematical expressions. However, many systems involving differential equations are so complex or the systems that they describe are so large that a purely analytical solution to the equation is not tractable.

It is in these complex systems where computer simulations and numerical approximations are useful. The techniques for solving differential equations based on numerical approximations were developed before programmable computers existed. The problem of solving ordinary differential equations is classified into two namely initial value problems and boundary value problems, depending on the conditions at the end points of the domain are specified. All the

conditions of initial value problem are specified at the initial point. There are numerous methods that produce numerical approximations to solution of initial value problem in ordinary differential equation such as Euler's method which was the oldest and simplest such method originated by Leonhard Euler in 1768, Improved Euler method, Runge Kutta methods described by Carl Runge and Martin Kutta in 1895 and 1905, respectively.

There are many excellent and exhaustive texts on this subject that may be consulted, such as Boyce and DiPrima (2001), Erwin (2003), Stephen (1983), Collatz (1960), and Gilat (2004) just to mention few. In this work we present the practical use and the convergence of Euler method for solving initial value problem in ordinary differential equation.

## NUMERICAL METHOD

The numerical method forms an important part of solving initial value problem in ordinary differential equation, most especially in cases where there is no closed form analytic formula or difficult to obtain exact solution. Next, we shall present Euler's method for solving initial value problems in ordinary differential equations.

### Euler's Method

Euler's method is also called tangent line method and is the simplest numerical method for solving initial value problem in ordinary differential equation, particularly suitable for quick programming which was originated by Leonhard Euler in 1768. This method subdivided into three namely:

- Forward Euler's method.
- Improved Euler's method.
- Backward Euler's method.

In this work we shall only consider forward Euler's method.

### Derivation of Euler's Method

We present below the derivation of Euler's method for generating, numerically, approximate solutions to the initial value problem:

$$y' = f(x, y(x)) \quad (1)$$

$$y(x_0) = y_0 \quad (2)$$

Where  $x_0$  and  $y_0$  are initial values for  $x$  and  $y$ , respectively. Our aim is to determine (approximately) the unknown function  $y(x)$  for  $x \geq x_0$ . We are told explicitly the value of  $y(x_0)$ , namely  $y_0$ , using the given differential equation (1), we can also determine exactly the instantaneous rate of change of  $y$  at point  $x_0$

$$y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0) \quad (3)$$

If the rate of change of  $y(x)$  were to remain  $f(x_0, y_0)$  for all point  $x$ , then  $y(x)$  would exactly  $y_0 + f(x_0, y_0)(x - x_0)$ . The rate of change of  $y(x)$  does not remain  $f(x_0, y_0)$  for all  $x$ , but it is reasonable to expect that it remains close to  $f(x_0, y_0)$  for  $x$  close to  $x_0$ . If this is the case, then the value of  $y(x)$  will remain close to  $y_0 + f(x_0, y_0)(x - x_0)$  for  $x$  close to  $x_0$ , for small number  $h$ , we have:

$$x_1 = x_0 + h \quad (4)$$

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)(x_1 - x_0) \\ y_1 &= y_0 + hf(x_0, y_0) \end{aligned} \quad (5)$$

Where  $h = x_1 - x_0$  and is called the step size. By the above argument:

$$y(x_1) \approx y_1 \quad (6)$$

Repeating the above process, we have at point  $x_1$  as follows:

$$x_2 = x_1 + h \quad (7)$$

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)(x_2 - x_1) \\ &= y_1 + hf(x_1, y_1) \end{aligned} \quad (8)$$

We have:

$$y(x_2) \approx y_2 \quad (9)$$

Then define for  $n = 0, 1, 2, 3, 4, 5, \dots$ , we have

$$x_n = x_0 + nh \quad (10)$$

Suppose that, for some value of  $n$ , we are already computed an approximate value  $y_n$  for  $y(x_n)$ . Then the rate of change of  $y(x)$  for  $x$  close to  $x_n$  is  $f(x, y(x)) \approx f(x_n, y(x_n)) \approx f(x_n, y_n)$  where  $y(x_n) = y_n + f(x_n, y_n)(x - x_n)$ .

Thus,

$$y(x_{n+1}) \approx y_{n+1} = y_n + hf(x_n, y_n) \quad (11)$$

Hence,

$$y_{n+1} = y_n + hf(x_n, y_n) \quad (12)$$

Equation (12) is called Euler's method. From (12), we have:

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n), n = 0, 1, 2, 3, \dots \quad (13)$$

### Truncation Errors For Euler's Method

Numerical stability and errors are discussed in depth in Lambert (1973) and Kockler (1994). There are two types of errors arise in numerical methods namely truncation error which arises primarily from a discretization process and round

off error which arises from the finiteness of number representations in the computer. Refining a mesh to reduce the truncation error often causes the round off error to increase. To estimate the truncation error for Euler's method, we first recall Taylor's theorem with remainder, which states that a function  $f(x)$  can be expanded in a series about the point  $x = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^m(a)(x-a)^m}{m!} + \frac{f^{m+1}(\beta)(x-a)^{m+1}}{(m+1)!} \quad (14)$$

The last term of (14) is referred to as the remainder term. Where  $x \leq \beta \leq a$ .

In (14), let  $x = x_{n+1}$  and  $x = a$ , in which:

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{1}{2}h^2 y''(\beta_n) \quad (15)$$

Where  $x_n \leq \beta_n \leq x_{n+1}$ .

Since  $y$  satisfies the ordinary differential equation in (1), which can be written as:

$$y'(x_n) = f(x_n, y(x_n)) \quad (16)$$

Where  $y(x_n)$  is the exact solution at  $x_n$ . Hence,

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)) + \frac{1}{2}h^2 y''(\beta_n) \quad (17)$$

By considering (17) to Euler's approximation in (12), it is clear that Euler's method is obtained by omitting the remainder term  $\frac{1}{2}h^2 y''(\beta_n)$  in the

Taylor expansion of  $y(x_{n+1})$  at the point  $x_n$ . The omitted term accounts for the truncation error in Euler's method at each step.

## Convergence of Euler's Method

The necessary and sufficient conditions for a numerical method to be convergent are stability and consistency. Stability deals with growth or decay of error as numerical computation progresses. Now we state the following theorem to discuss the convergence of Euler's method.

**Theorem:** If  $f(x, y)$  satisfies a Lipschitz condition in  $y$  and is continuous in  $x$  for  $0 \leq x \leq a$  and defined a sequence  $y_n$ , where  $n = 1, 2, \dots, k$  and  $y_0 \rightarrow y(0)$ , then  $y_n \rightarrow y(x)$  as  $n \rightarrow \infty$  uniformly in  $x$  where  $y(x)$  is the solution of the initial value problem (1) and (2).

**Proof:** we shall start the proof of the above theorem by deriving a bound for the error:

$$e_n = y_n - y(x_n) \quad (18)$$

Where  $y_n$  and  $y(x_n)$  are called numerical and exact values respectively. We shall also show that this bound can be made arbitrarily small. If a bound for the error depends only on the knowledge of the problem but not on its solution  $y(x)$ , it is called an a priori bound. If, on the other hand, knowledge of the properties of the solution is required, its error bound is referred to as an a posteriori bound.

To get an a priori bound, let us write:

$$y(x_{n+1}) = y(x_n) + hf(x_n, y_n) - t_n \quad (19)$$

Where  $t_n$  is called the local truncation error. It is the amount by which the solution fails to satisfy the difference method. Subtracting (19) from (12), we get:

$$e_{n+1} = e_n + h[f(x_n, y_n) - f(x_n, y(x_n))] + t_n \quad (20)$$

Let us write:

$$e_n M_n = f(x_n, y_n) - f(x_n, y(x_n)) \quad (21)$$

Substituting (20) into (21), then:

$$e_{n+1} = e_n(1 + hM_n) \quad (22)$$

This is the difference equation for  $e_n$ . The error  $e_0$  is known, so it can be solved if we know  $M_n$  and  $t_n$ . We have a bound of the Lipschitz constant  $M$  for  $|M_n|$ . Suppose we also have  $T \geq |t_n|$ . Then we have:

$$|e_{n+1}| \leq |e_n|(1 + hM) + T \quad (23)$$

To proceed further, we need the following lemma.

**Lemma:** If  $|e_n|$  satisfies (23) and  $0 \leq nh \leq a$ , then:

$$\begin{aligned} |e_n| &= T \frac{(1 + hM)^n - 1}{hM} + (1 + hM)^n |e_0| \\ &\leq \frac{T}{hM} (e^{Lb} - 1) + e^{Lb} |e_0| \end{aligned} \quad (24)$$

**Lemma:** The first inequality of (24) follows by induction. It is trivially true for  $n = 0$ . Assuming that it is true for  $n$ , we have from (23):

$$\begin{aligned} |e_{n+1}| &= T \frac{(1 + hM)^{n+1} - 1}{hM} + (1 + hM)^{n+1} |e_0| \\ &= T \frac{(1 + hM)^{n+1} - 1}{hM} + (1 + hM)^{n+1} |e_0| \end{aligned} \quad (25)$$

Hence (24) is true for  $n + 1$  and thus for all  $n$ . The second inequality in (25) follows from the fact that  $nh \leq a$  and for  $hM \geq 0$ ,  $(1 + hM) \leq e^{Mh}$  so that  $(1 + hM)^n \leq e^{Mnh} \leq e^{Ma}$ , proving the lemma. To continue the proof of the theorem, we need to investigate  $T$ , the bound on the local truncation error.

From (19), we have:

$$-t_n = y(x_{n+1}) - y(x_n) - hf(x_n, y(x_n))$$

$$\begin{aligned} &\text{By the Mean value theorem, we get for } 0 \leq \theta \leq 1, \\ &\leq h|f(x_n + \theta h, y(x_n)) - f(x_n, y(x_n))| + \\ &h|f(x_n + \theta h, y(x_n + \theta h)) - f(x_n + \theta h, y(x_n))| \\ &\leq h|f(x_n + \theta h, y(x_n)) - f(x_n, y(x_n))| + \\ &h|y(x_n + \theta h) - y(x_n)| \end{aligned} \quad (26)$$

The last term can be treated by the Mean value theorem to get a bound  $M\theta h^2|y'(g)| \leq h^2MZ$ , where

$Z = \max|y'(x)|$ , the inequality exists because of the continuity of  $y$  and  $f$  in a closed region. The treatment of the first term in (26) depends on our hypothesis. If we are prepared to assume that  $f(x, y)$  also satisfies a Lipschitz condition in  $x$ , we can bound the first term in (26) by  $L\theta h^2$ , where  $L$  is the Lipschitz constant for  $f(x)$ . Consequently,  $|t_n| \leq h^2(L + MZ) = T$  and so from (24), we get:

$$|e_n| \leq h \frac{L + MZ}{M} (e^{Ma} - 1) + e^{Ma} |e_0| \quad (27)$$

Thus the numerical solution converges as  $h \rightarrow 0$ , if  $|e_0| \rightarrow 0$ .

### **Algorithm for Euler's Method (Samuel, 1981)**

The typical steps of Euler's method are given below:

Step 1: define  $f(x_n, y_n)$

Step 2: input initial values  $x_0$  and  $y_0$

Step 3: input step sizes  $h$  and number of steps  $n$

Step 4: calculate  $x$  and  $y$

$$\text{for } n = 1 : N - 1$$

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Step 5: output  $x_{n+1}$  and  $y_{n+1}$

Step 6: end

### NUMERICAL EXAMPLES

Now, we present some numerical examples as follows:

**Example 1:** A ball at  $927^{\circ}C$  is allowed to cool down in air at an ambient temperature of  $27^{\circ}C$ . Assuming heat is lost only due to radiation. The differential equation for the temperature of the ball is given by:

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 8.1 \times 10^9), \theta(0) = 927^{\circ}C$$

Find the temperature at  $t = 480s$ . Assume a step size of  $h = 240s$ .

Step 1:

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}(\theta^4 - 8.1 \times 10^9), t_0 = 0,$$

$$\theta_0 = 1200k$$

$$f(t, \theta) = -2.2067 \times 10^{-12}(\theta^4 - 8.1 \times 10^9),$$

$$\theta_{n+1} = \theta_n + hf(t_n, \theta_n)$$

$$\theta_1 = \theta_0 + hf(t_0, \theta_0)$$

$$\begin{aligned} \theta_1 &= 1200 + 240f(0, 1200) \\ &= 1200 + 240 \times (1200^4 - 8.1 \times 10^9)(-2.2067 \times 10^{-12}) \\ &= 1200 + (-4.5574)(240) = 106.09k \end{aligned}$$

Therefore,

$$\theta_1 \approx \theta(t_1), t_1 = t_0 + h = 0 + 240 = 240$$

$$\theta(240) = 106.09k \approx \theta_1$$

Step 2: For  $n = 1, t_1 = 240, \theta_1 = 106.09$

$$\theta_2 = \theta_1 + hf(t_1, \theta_1)$$

$$\begin{aligned} \theta_1 &= 106.09 + 240f(240, 1200) \\ &= 1200 + 240 \times (106.09^4 - 8.1 \times 10^9)(-2.2067 \times 10^{-12}) \\ &= 106.09 + (0.017545)(240) = 110.32k \end{aligned}$$

$$\theta_2 \approx \theta(t_2), t_2 = t_1 + h = 240 + 240 = 480$$

$$\theta(480) = 110.32k \approx \theta_2$$

The exact solution of the ordinary differential equation is given by the solution of a non linear equation as:

$$\begin{aligned} 0.92593 \ln\left(\frac{\theta - 300}{\theta + 300}\right) - 1.8519 \tan^{-1}(0.00333\theta) = \\ -0.22067 \times 10^{-3}t - 2.9282 \end{aligned}$$

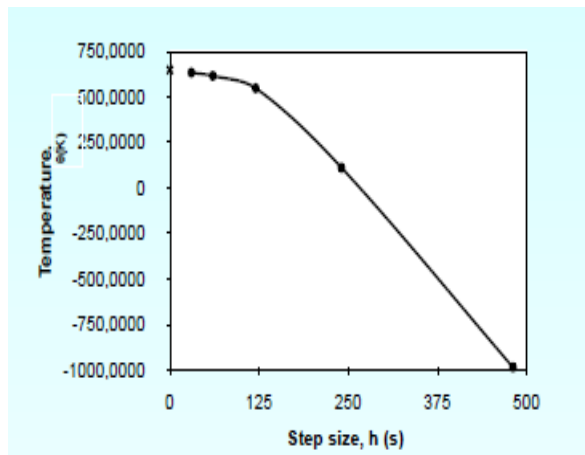
The solution to this nonlinear equation at  $t = 480s$  is  $\theta(480) = 647.57k$ . The effect of step size on Euler's method is shown in Table 1 and Figure 1.

**Table 1:** Effect of Step Size (Temperature at 480seconds as a function of step size,  $h$ ).

Step Size, $h$	$\theta(480)$	$E_t$	$ \epsilon_t $
60	546.77	32.607	5.0352
120	110.32	100.80	15.566
240	110.32	537.26	82.964
480	-987.8	1635.4	252.54

**Example 2:** We use Euler's method to approximate the solution of the initial value problem  $y' = y - x, y(0) = 2$ , with step size  $h = 0.1$  on the interval  $0 \leq x \leq 1$  whose exact solution is given by  $y(x) = e^x + x + 1$ . The results obtained shown in Table 2, the comparison of the method to the exact solution and the error incurred in Euler's method.

**Figure 1:** Effect of Step Size on Euler's Method



**Table 2:** The Comparative Result Analysis and Error generated from Euler's Method.

$n$	$x_n$	$y_n$	$y(x_n)$	$e_n = y(x_n) - y_n$
0	0.0	2.0000	2.0000	0.0000
1	0.1	2.2000	2.2052	0.0052
2	0.2	2.4100	2.4214	0.0114
3	0.3	2.6310	2.6498	0.0188
4	0.4	2.8641	2.8918	0.0277
5	0.5	3.1105	3.1487	0.0382
6	0.6	3.3716	3.4221	0.0505
7	0.7	3.6487	3.7137	0.0650
8	0.8	3.9436	4.0255	0.0819
9	0.9	4.2579	4.3596	0.1017
10	1.0	4.5937	4.7182	0.1245

**Example 3:** We shall approximate the solution of the initial value problem  $y' = 1 + y^2$ ,  $y(0) = 2$  using Euler's method with step size  $h = 0.1$  on the interval  $0 \leq x \leq 1$  whose exact solution is given by  $y(x) = \tan x$ . The results obtained shown in Table 3, the comparison of the method to the exact solution and the error incurred in Euler's method.

### DISCUSSION OF RESULTS

We notice that in example 1, the accuracy of the approximations gets worse as we further away from the initial value and in examples 2 and 3, the error get larger as  $n$  increases.

**Table 3:** The Comparative Result Analysis and Error generated from Euler's Method.

$n$	$x_n$	$y_n$	$y(x_n)$	$e_n = y(x_n) - y_n$
0	0.0	0.0000	0.0000	0.0000
1	0.1	0.1000	0.1003	0.0003
2	0.2	0.2010	0.2027	0.0017
3	0.3	0.3050	0.3093	0.0043
4	0.4	0.4143	0.4227	0.0084
5	0.5	0.5315	0.5463	0.0148
6	0.6	0.6598	0.6841	0.0243
7	0.7	0.8033	0.8422	0.0389
8	0.8	0.9678	1.0296	0.0618
9	0.9	1.1615	1.2601	0.0986
10	1.0	1.3964	1.5574	0.1610

### CONCLUSION

In general, each numerical method has its own advantages and disadvantages of use: Euler's method is therefore best reserved for simple preferably, recursive derivatives that can be represented by few terms. It is simple to implement and simplifies rigorous analysis. The major disadvantages of this method are the tiresome, sometimes impossible calculation of higher derivatives and the slow convergence of the series for some functions which involves terms of opposite sign.

### REFERENCES

1. Boyce, W.E. and R.C. DiPrima. 2001. *Elementary Differential Equation and Boundary Value Problems*. John Wiley and Sons: New York, NY.
2. Collatz, L. 1960. *Numerical Treatment of Differential Equations*. Springer Verlag: Berlin, Germany.
3. Erwin, K. 2003. *Advanced Engineering Mathematics. Eighth Edition*. Wiley Publishers: New York, NY.
4. Gilat, A. 2004. *Matlab: An Introduction with Application*. John Wiley and Sons: New York, NY.
5. Kockler, N. 1994. *Numerical Methods and Scientific Computing*. Clarendon Press: Oxford, UK.
6. Lambert, J.D. 1991. *Numerical Method for Ordinary Systems of Initial Value Problems*. John Wiley and Sons: New York, NY.
7. Samuel, D.C. 1981. *Elementary Numerical Analysis: An Algorithm Approach. Third Edition*. Mc Graw International Book Company: New York, NY.

8. Stephen, M.P. 1983. *To Compute Numerically, Concepts and Strategy*. Little Brown and Company. Ottawa, Canada.

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