

The Development of Image-Conjugate Theory for the Stabilization of Unstable Linear Control Systems.

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ABSTRACT

Location of poles of a characteristic equation of a closed-loop control system in the complex s-plane, are illustrated. Stability of control systems, relative to these locations, is discussed. A typical unstable control system, as a result of its poles locations, is presented.

The pole location which contributes to the instability is moved to its image position in the left-half of the complex s-plane. This new position of the pole, which initially contributed to instability, is used to determine the forward gain of the control system. The new forward gain is different from the previous one. For this system to be stable, it has to be re-designed to have this new forward gain. Components of the new control system would vary, giving it new dynamics, resulting in the stability of the system.

(Keywords: systems, poles, complex-plane, stability, image, conjugate)

INTRODUCTION

If the real or complex conjugate poles of the characteristics equation of a closed-loop linear control system lie on the right-half of the complex s-plane, such a control system is said to be unstable otherwise it is not (Nagrath and Gopal, 2009). The pole locations of such control systems, and their respective time output responses, are illustrated in Figure 1.

In Figure 1(b), the time output response is a growing exponential function with time while that of Figure 1(d) is a growing oscillatory function with time. Both responses indicate unstable control systems, as already mentioned. Though these types of systems are unstable, this paper

proposes a design technique of how to stabilize such systems by considering the equivalent location of those poles in the left-half of the complex s-plane; their “images” in the left-half plane.

THE CONCEPT OF “IMAGE” IN ELECTRICAL ENGINEERING

In Electrical Engineering, an image of a point is the replica of that point in the opposite side of a referenced plane (Morton, 2010). By this definition, considering the $j\omega$ axis as the plane of reference, the image location of the pole a_1 , will be as illustrated in Figure 2(a). Correspondingly, its typical time output response graph is illustrated in Figure 2(b).

Systems with pole locations and time output responses as shown in Figures 2(a) and 2(b) respectively, are said and known to be stable (Nagrath and Gopal, 2009).

Similar to Figures 2(a) and 2(b), considering the $j\omega$ axis as the plane of reference, image positions for Figures 1(c) and 1(d) will be as illustrated below.

Systems with pole locations, and time output responses, as shown in Figures 2(c) and 2(d) respectively, are equally said and known to be stable (Nagrath and Gopal, 2009). In Figure 2(b), the time output response is a decaying exponential function with time while that of Figure 2(d) is a decaying oscillatory function with time. Both responses indicate stable control systems, as already mentioned.

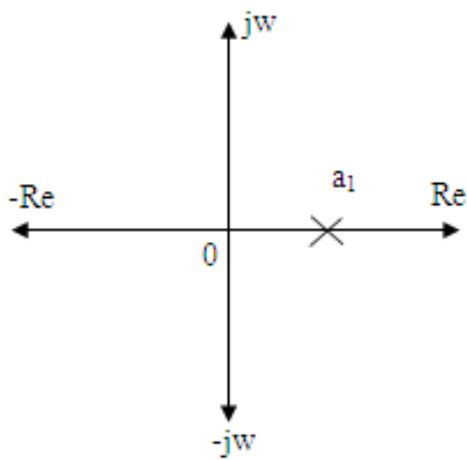


Figure 1(a): Single Real Pole Location in the right-half Complex S-Plane.

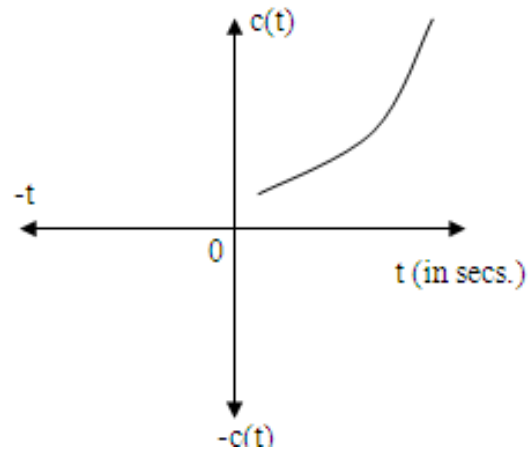


Figure 1(b): A Typical Time Output Response of a Control System with Pole Location shown in Figure 1(a).

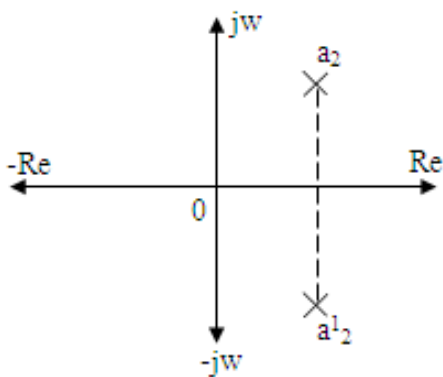


Figure 1(c): A Complex Conjugate Pole Location in the right-half of the Complex S-Plane.

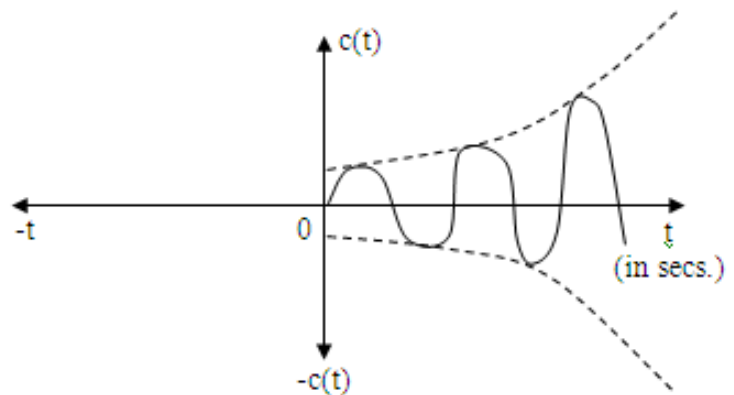


Figure 1(d): A Typical Time Output Response of a Control System with Pole Location Shown in Figure 1(c).

Considering these scenarios, suppose after solving for the roots of the characteristic equation of a negative feedback closed-loop control system, and placing them where they belong on the complex s-plane, one finds that they are on the right-half of the plane, by the analyses done

so far, it would be concluded immediately that such system is unstable. Yes, this is true but this paper attempts to make such system stable by “backward calculation” using the proposed principle of image – conjugate theory, as will be shown next.

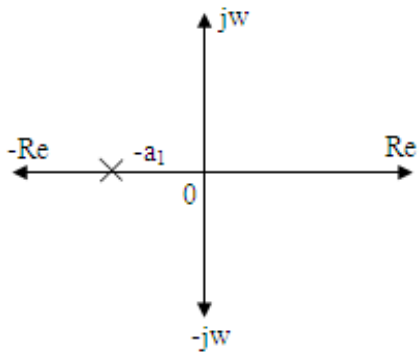


Figure 2(a): Image Location of Pole a_1 Shown in Figure 1(a).

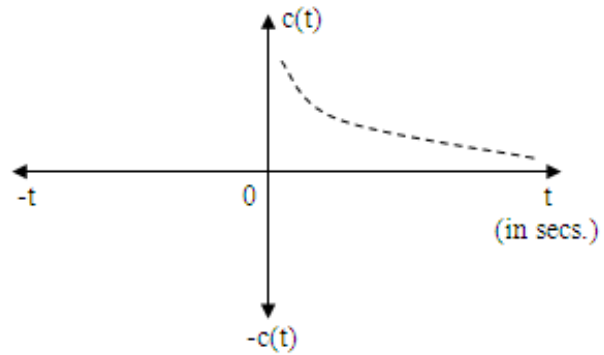


Figure 2(b): A Typical Time Output Response for a System with Pole Location Shown in Figure 2(a).

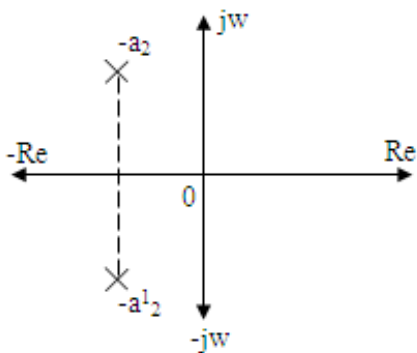


Figure 2(c): Image Locations of Complex Conjugate Poles shown in Figure 1(c).

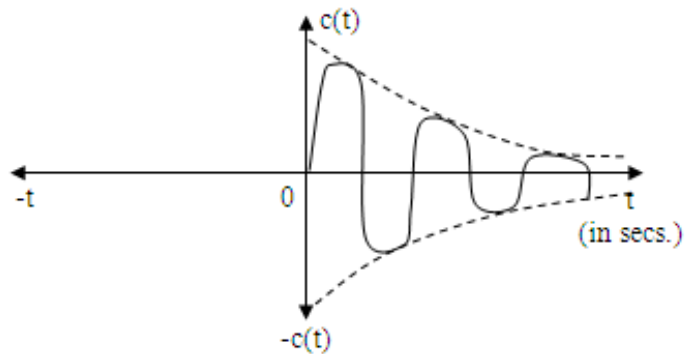


Figure 2(d): A Typical Time Output Response of a Control System with Pole Location Shown in Figure 2(c).

STABILIZING AN UNSTABLE LINEAR NEGATIVE FEEDBACK CONTROL SYSTEM THROUGH THE IMAGE-CONJUGATE THEORY

This will be best attempted with a referenced unstable linear negative feedback control system presented below.

“Determine if the negative unity feedback control system with a characteristic equation given by: $s^3 - 7s - 6 = 0$, is stable or not”.

(Distefano, Stubberud and William, 2010).

Solution

First of all, the characteristic equation has to be factorized. Hence we proceed by the trial and error approach, as follows, to determine the factors:

$$s^3 - 7s - 6 = 0 \quad \dots \quad (1)$$

We try $s = -1$

Equation (1) hence becomes:

$$-1 + 7 - 6 = 0 \quad \dots \quad (2)$$

This implies that $s + 1$ is a factor. Next, we try $s = -2$

Equation (1) becomes:

$$-8 + 14 - 6 = 0 \dots \quad (3)$$

This implies that $s + 2$ is a factor. To determine the third factor, we proceed by dividing the given expression by the product of $(s + 1)$ and $(s + 2)$. That is:

$$\frac{s^3 - 7s - 6}{(s + 1)(s + 2)} = \frac{s^3 - 7s - 6}{s^2 + 3s + 2}$$

which is:

$$s^2 + 3s + 2 \begin{array}{r} s - 3 \\ \hline s^3 - 7s - 6 \\ s^3 + 3s^2 + 2s \\ \hline -3s^2 - 9s - 6 \\ -3s^2 - 9s - 6 \\ \hline 0 \quad 0 \quad 0 \end{array}$$

That means the third factor is $(s - 3)$.

$$\therefore s^3 - 7s - 6 = (s + 1)(s + 2)(s - 3) \dots \quad (4)$$

In other words, the factors of $s^3 - 7s - 6$ are $-1, -2, 3$. Hence, the poles of the characteristic equation: $(-1, -2, 3)$, and their locations in the complex s -plane, are as shown in Figure 3(a).

Since one of the poles is in the right half of the complex s -plane, the system is unstable. The

“offending” pole is “3” which is located in the right-half of the s -plane.

The forward gain, $G(s)$ of the given system is determined from the closed-loop transfer function as follows:

$$\frac{G(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \dots \quad (5) \quad (\text{Ogata, 2008})$$

Where, $R(s)$ = the input to the system, $C(s)$ = the output of the system and, $H(s)$ = the feedback gain.

The characteristic equation of Equation (5) is given by:

$$1 + G(s)H(s) = 0 \dots \quad (6)$$

but,

$$1 + G(s)H(s) = s^3 - 7s - 6 \quad (\text{referring to the given problem}) \dots \quad (7)$$

and $H(s) = 1$.

Hence Equation (7) becomes:

$$1 + G(s) = s^3 - 7s - 6$$

or

$$G(s) = s^3 - 7s - 7 \dots \quad (8)$$

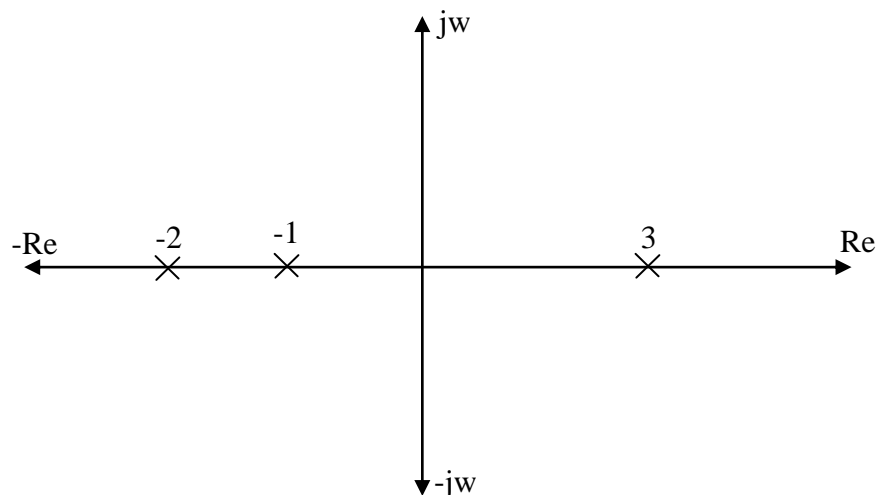


Figure 3(a): Location of Poles of Equation (1) in the Complex S-Plane.

This means that the system could be represented in a closed-loop configuration as illustrated in Fig. 3(b) below:

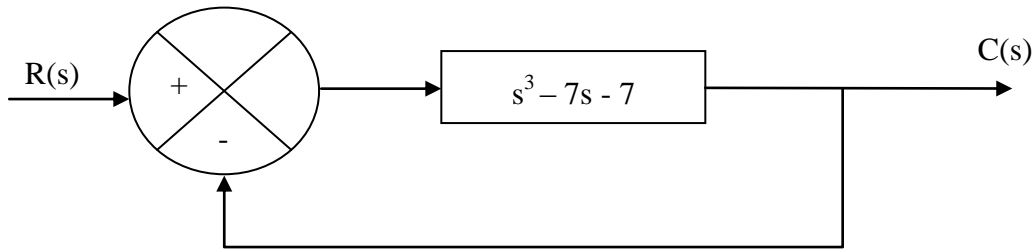


Figure 3(b): A Closed-Loop Configuration of the Given Control System Showing its Forward Gain, G(s).

From Equation (5):

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s) H(s)} = \frac{s^3 - 7s - 7}{1 + (s^3 - 7s - 7) 1}$$

$$\frac{C(s)}{R(s)} = \frac{s^3 - 7s - 7}{s^3 - 7s - 6} \quad (9)$$

THE IMAGERY OF THE OFFENDING POLE LOCATION IN THE STABILITY OF THE GIVEN SYSTEM

If the “3” is moved to its image position in the left-half of the complex s-plane, then all the poles will be in the left-half of the plane. The pole locations would then be as shown in Figure 4(a).

This movement of pole from the right half to the left half of the complex s- plane could cause some changes in the original system. This is because; in as much as we are trying to obtain a stable control system, the system gain and dynamics could change considerably. For the system under consideration, after the movement of the offending pole to its image position in the left- half complex s-plane, the characteristic equation becomes:

$$(s + 1)(s + 2)(s + 3) = 0 \dots \quad (10)$$

which leads to:

$$s^3 + 6s^2 + 11s + 6 = 0 \dots \quad (11)$$

Equation (11) implies that:

$$1 + G(s) H(s) = s^3 + 6s^2 + 11s + 6 \dots \quad (12)$$

Since H(s) = 1, Equation (12) becomes:

$$1 + G(s) = s^3 + 6s^2 + 11s + 6$$

$$G(s) = s^3 + 6s^2 + 11s + 5 \dots \quad (13)$$

From which,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s) H(s)} = \frac{s^3 + 6s^2 + 11s + 5}{s^3 + 6s^2 + 11s + 6} \quad (14)$$

The closed-loop configuration of the new system is illustrated in Figure 4(b).

DISCUSSION

It can be seen that the forward gains of the two systems, that of Figures 3 (b) and (4c) are different. In practical terms, what this means to the systems engineer is:

“If you want the control system whose characteristic equation is given in equation (1) to be stable, go back and re-design the system such that the forward gain of the system becomes: $s^3 + 6s^2 + 11s + 5$ instead of $s^3 - 7s - 7$ ”.

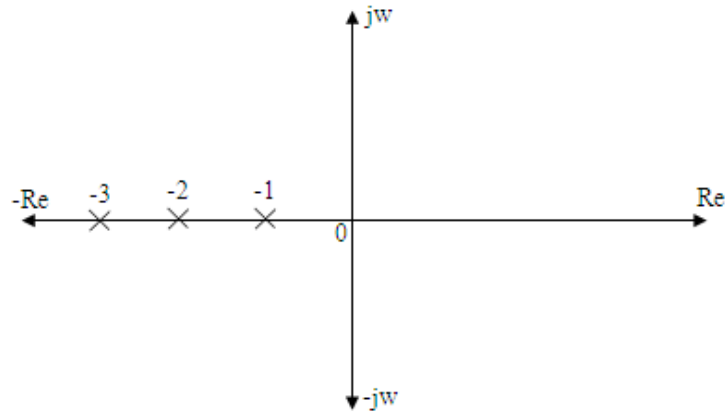


Figure 4 (a): The Pole Contributing to Instability Moved to its Image Position in the Complex S-Plane.

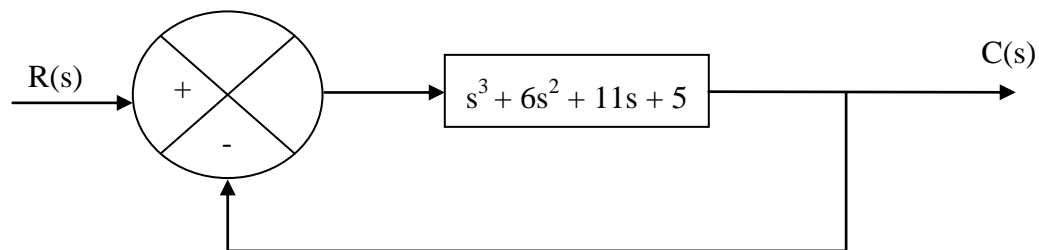


Figure 4 (c): A Closed-Loop Configuration of the New System showing its Forward Gain, $G(s)$.

That means that the components of the original system were wrong or poorly selected. This concept of system design can be applied to any linear feedback control system either with real or complex conjugate roots. If the roots of the characteristic equation (the poles) of a control system are known, then this theory becomes a tool for designing a stable control system.

CONCLUSION

The “image-conjugate theory,” presented in this paper, is to serve as a design tool for control engineers. It is developed from the location of poles of the characteristic equation in the complex s-plane. It is hoped that this concept of systems design would complement the existing design theories of control systems for stability.

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