Toward a Predator-Prey System of Holling Type.

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ABSTRACT

This paper considers the linearized form of the predator-prey system of Holling type and discusses the existence and non-existence of limit cycle using the properties of the Jacobian matrix slopes of the slopes sepatrices of the system. The main result is a criterion derived for the stability non-stability of equilibrium and the criterion is based on Freedman's (1980) criteria for the stability of equilibria for a generalized Gauss model. Examples are presented to illustrate an agreement between our result and the results of Sugie *et al.* (1997).

(Keywords: linearized predator prey system, Holling Type, limit cycle, Jacobian matrix slopes, equilibrium)

INTRODUCTION

The objective of this paper is to contribute toward the uniqueness of limit cycle for a predator-prey system of Holling Type. More specifically, we shall consider the system of the form:

$$x' = rx\left(1 - \frac{x}{k}\right) - \frac{x^p y}{a + x^p}$$

$$y' = y\left(\frac{\mu x^p}{a + x^p} - D\right), \qquad \textbf{(1)}$$

where x and y represent the prey and the predator population, respectively; r, k, a, μ, D and p are positive parameters. The function $\frac{x^p}{a+x^p}$ is often called a functional response of Holling Type, when $p = 1$ or $p = 2$. System (1) has been studied by quite a number of authors like Cheng (1981), and Grasull and Guillamon (1993).

Other authors who also carried out an investigation of the system (1) for the uniqueness of limit cycle include Sugie *et al* (1996), Sugie *et al.* (1997) and Sugie (1998).

Also Koiji and Zegiling (1996) studied a predator-prey system of Ivlev functional response while Sesay *et al.* (2010) studied recently a predator-prey system of Ivlev functional response via the Jacobian matrix of the linearized system. Kuang and Freedman (1988) studied the uniqueness of limit cycle in Gause type model of predator-prey system by transforming the system into a Liѐnard's type.

Attili and Mallak (2006) considered a predator-prey system with the functional response of the form $\varphi(x) = arctan(ax); a > 0$. The main concern is the existence of limit cycles for such system. A necessary and sufficient condition for the nonexistence of limit cycles is given for such a system.

In the study of predator-prey model with Ivlev functional response and impulsive perturbations, Baek (2007) proves that there exists a stable prey-free solution when the impulsive period is less than the critical value. Also, he finds a sufficient condition that the model is permanent.
In another study by Beak (2008) of a predator-prey system with Michealis-Menten functional response and impulsive perturbations, which contains chemical and biological control terms, he applied the Floquet theory and establishes condition for the existence and stability of prey free solutions of the system.

This paper is mainly concerned with establishing necessary and sufficient condition for the existence and nonexistence of limit cycle for a predator-prey system of Holling type. Our work is based on Sugie *et al.* (1997) who proved and extended the work of Sugie *et al.* (1996). First we linearize system (1) to get the eigenvalues of the Jacobian matrix and use the method of isoclines to confirm the result of linearization by relating the signs of the eigenvalues to signs of the slope of the isoclines.

The main result is the derivation of a criterion on the existence and nonexistence of limit cycle for system (1), the criterion is an analogue of freedman's criterion for the stability of equilibriua of the generalized Gauss model. Examples are given for *p=1 and p=2* to illustrate that our result agree with theorems in Sugie *et* al (1997).

It is assumed in Sugie *et al* (1997) that, if $\mu > D$

and $k > \lambda_p = \sqrt[p]{\frac{aD}{\mu - D}}$, **(2)**

the system (1) has the only equilibrium point (λ_p, γ_p) in the first quadrant $\{(x, y) : x > 0,\ y > 0\}$,where $\gamma_p = \frac{r\mu}{D}\left(1 - \frac{\lambda_p}{k}\right)\lambda_p$.

The significance of this assumption (2) is seen in the following Theorems and a Conjecture (Sugie *et al.*, 1997).

Theorem A

Let $p = 1$. Then under assumption (2), system (1) has a unique stable limit cycle if and only if $(D + \mu)\lambda_1 < Dk$.

Theorem B

Let $p = 2$. Then, under assumption (2) system (1) has a unique stable limit cycle if and only if $2D\lambda_2 < (2D - \mu)k$.

Conjecture C

Let p be any positive integer, then under assumption (2), system (1) has no limit cycle if and only if:

$(pD - (p - 2))\lambda_p \geq (pD - (p - 1)\mu)k$. **(3)**

The proof that this conjecture is true for a positive integer and its extension for p any real number satisfying a certain condition are in Sugie *et al.* (1997). More precisely, if condition (3) holds, then system (1) has no limit cycles; otherwise, system (1) has unique limit cycle.

Linearization of System

In this section, we linearize the system (1) and relate condition (3) to the stability of the equilibrium (λ_p, γ_p) of system (1). In order to achieve this, we obtain the Jacobian matrix J at the equilibrium point:

$\left(\lambda_p = \left(\frac{aD}{\mu - D}\right)^{\frac{1}{p}}, \gamma_p = \frac{Dr}{\mu}\lambda_p\left(1 - \frac{\lambda_p}{k}\right)\right)$ as

$$J = \begin{bmatrix} \frac{r}{\mu k}[((1 - p)\mu + pD)k + ((p - 2)\mu - pD)\lambda_p] & -\frac{D}{\mu} \\ pr(\mu - D)\left(1 - \frac{1}{k}\lambda_p\right) & 0 \end{bmatrix}$$

The trace of J is:

$$\tau = \frac{r}{\mu k}[(1 - p)\mu + pD]k + [(p - 2)\mu - pD]\lambda_p$$

and its determinant is:

$$\delta = \frac{D}{\mu k} pr(\mu - D)(k - \lambda_p).$$

The eigenvalues of J can be determined from its characteristic equation $\sigma^2 - \sigma\tau + \delta = 0$, the solutions of which are $\sigma_1 = \frac{\tau + \sqrt{\tau^2 - 4\delta}}{2}$ and $\sigma_2 = \frac{\tau - \sqrt{\tau^2 - 4\delta}}{2}$.

An analysis of the dynamics of (1) and the stability of equilibrium point (λ_p, γ_p) starts in this section and continues in the subsequent sections.

Case 1: The Jacobian matrix J possesses eigenvalues having real parts with opposite signs. In this case, the discriminant $\tau^2 - 4\delta$ is positive and the determinant δ of J is negative. The equilibrium point (λ_p, γ_p) is an unstable saddle point, since $\sigma_1\sigma_2 = 4\delta < 0$. We conclude that system (1) has no limit cycle.

Case 2: The Jacobian matrix J possesses eigenvalues having negative real parts, and, in this case, the discriminant $\tau^2 - 4\delta > 0$, trace τ is negative and the determinant δ of J is positive. The equilibrium point (λ_p, γ_p) is stable node. Again, the conclusion is that system (1) has no limit cycle.

Case 3: The Jacobian matrix J possesses eigenvalues having real parts that are both positive. In this case, the discriminant $\tau^2 - 4\delta > 0$, trace $\tau > 0$ and the determinant $\delta > 0$. The equilibrium point (λ_p, γ_p) is an unstable node. From this case, we observe that a necessary and sufficient condition for system (1) to possess a unique stable limit cycle is when $\tau > 0$. This is essentially the same result which Sugie *et al.* (1997) established by utilizing the properties of Liènard's equation.

Non-Existence of Limit Cycle

We take first the case for the absence of limit cycles. Case 1 states that J possesses eigenvalues with real parts having opposite signs. Here, we deduce (3) in the introduction of this paper from this case as outlined below.

In this case $\tau^2 - 4\delta > 0$, and $\delta < 0$. The left hand sides of the two inequalities represent the discriminant of the characteristic equation of J and determinant, respectively. Trace of J is:

$$\tau = \frac{r}{\mu k}[(1-p)\mu + pD]k + [(p-2)\mu - pD]\lambda_p$$

and determinant is $\delta = \frac{pDr}{\mu}(\mu - D)\left(1 - \frac{\lambda_p}{k}\right)$.

Suppose that $\tau < 0$, then (3) follows. Also, $\delta < 0$ implies that $\delta = \frac{pDr}{\mu}(\mu - D)\left(1 - \frac{\lambda_p}{k}\right) < 0$. This further implies that $k < \lambda_p = \left(\frac{aD}{\mu - D}\right)^{\frac{1}{p}}$ and hence assumption (2) does not hold. Combining (2) and (3), it follows that system (1) has no limit cycle.

Remark 1: In case 2, we observe that the trace of J is also negative implying that (3) is also satisfied. However, the determinant is positive implying that $k > \lambda_p$. This means that the second inequality in assumption (2) is not sufficient to guarantee the existence of limit cycle.

Remark 2: The conjecture C in Sugie *et al.* (1997), that is, the inequality (3) can be easily deduced from the case when $\tau \leq 0$. Even if the trace is positive and the determinant is negative, there is no limit cycle.

Remark 3: When assumptions (2) and (3) are both satisfied, and p is a positive number with $p \leq \frac{1}{2}$ or $p \geq 1$, then system (1) has no limit cycles. The proof of this may be found in Sugie *et al.* (1997). They have also proved a condition for system (1) to have limit cycle if $p \leq 1$ or $p \geq 2$. The condition can be established utilizing $\tau > 0$ as in case 3.

The existence of an equilibrium point in the first quadrant and regions where it can possibly exist (or not exist) is important because it is related to the assumption that $\delta \neq 0$. When $\delta = 0$, there is an equilibrium point $(k, 0)$ on the x axis and the line $x = k$, passing through this point $(k, 0)$, separates points in the first quadrant into two regions: one region is left of the line $x = k$, where one can possibly find an equilibrium point, and the other region is right of the line, where equilibrium points do not exist.

The Jacobian matrix at $(k.0)$ reduces to:

$$J(k,0) = \begin{bmatrix} -r & \dfrac{-k^p}{a + k^p} \\ 0 & \dfrac{\mu k^p}{a + k^p} - D \end{bmatrix}$$

And its trace becomes $\tau = \frac{\mu k^p}{a + k^p} - (r + D)$ and the determinant $\delta = -r\left(\frac{\mu k^p}{a + k^p} - D\right)$. The eigenvalues of $J(k, 0)$ are given by the roots $(\sigma_1$ and $\sigma_2)$ of the equation $\sigma^2 - \tau\sigma + \delta = 0$, where $\sigma_1 = -r$ and $\sigma_2 = \frac{\mu k^p}{a + k^p} - D$.

One can easily show that the point $(k, 0)$ is an unstable saddle point. In order to do this, suppose that σ_2 does vanish then $k = \left(\frac{aD}{\mu - D}\right)^{\frac{1}{p}}$ and assumption (2) does not hold. So σ_2 does not vanish if (2) is to hold.

Also, σ_2 cannot be less than zero, for if it is, then $k < \left(\frac{aD}{\mu - D}\right)^{\frac{1}{p}}$ and this inequality suggests that (2) will be violated. From this we conclude that $\sigma_2 > 0$, i.e, (2) holds when σ_2 is positive. Since $\sigma_1 = -r < 0$, it follows that $(k, 0)$ is an unstable saddle point.

METHOD OF ISOCLINES

The isoclines for system (1) are given by the equations:

$$\frac{dy}{dx} = \frac{y\left(\frac{\mu x^p}{a + x^p} - D\right)}{rx\left(1 - \frac{x}{k}\right) - \frac{\mu x^p}{a + x^p}} = c, \qquad \textbf{(4)}$$

where c is a constant. We express these as the family of curves $y = \frac{crx\left(1 - \frac{x}{k}\right)}{\left(\frac{x^p}{a + x^p}\right)(\mu - c) - D}$.

From the second equality in (4), we see that, $y\left(\frac{\mu x^p}{a + x^p} - D\right) = c\left[rx\left(1 - \frac{x}{k}\right) - \frac{\mu x^p}{a + x^p}\right]$.

Now let
$h(x, y, c) = y\left(\frac{\mu x^p}{a + x^p} - D\right) - c\left[rx\left(1 - \frac{x}{k}\right) - \frac{\mu x^p}{a + x^p}\right]$.

The slope of the separatrices at (λ_p, γ_p) are determined by solving:

$$\frac{\partial h}{\partial x}\left(\lambda_p, \gamma_p, c\right) + c\frac{\partial h}{\partial y}\left(\lambda_p, \gamma_p, c\right) = 0. \qquad \textbf{(5)}$$

The partial derivatives of h are computed at (λ_p, γ_p) and on simplification, (5) leads to the equation:

$$kDc^2 - cr\left[\left((1 - p)\mu + pD\right)k + \left((p - 2)\mu - pD\right)\lambda_p\right] + r\mu p(\mu - D)\left(k - \lambda_p\right) = 0 \qquad \textbf{(6)}$$

Equation (6) is further simplified to:

$$kDc^2 - c(\mu k\tau) + \left(\frac{\mu^2 k\delta}{D}\right)\mu = 0, \qquad \textbf{(7)}$$

using τ and δ as stated in section 2.

Solving this quadratic equation for c gives the values of c as:

$$c = \frac{(\mu k)\tau \pm \sqrt{\left((\mu k)\tau\right)^2 - 4(\mu k)^2\delta}}{2kD}$$

The two solutions to equation (7), that is, $(c_1 \ and \ c_2)$ are the slopes of the separatrices at the equilibrium point (λ_p, γ_p), their signs coincide with those of $\delta_1 \ and \ \delta_2$ of the characteristic equation of J, i.e, $\sigma^2 - \tau\sigma + \delta = 0$. The three cases in section 2 can now be confirmed using the slopes of the separatrices $c_1 \ and \ c_2$.

1. When $c_1 \ and \ c_2$ are of opposite signs, the equilibrium point (λ_p, γ_p) is an unstable saddle point.

2. When $c_1 \ and \ c_2$ possess negative real parts, the equilibrium point (λ_p, γ_p) is a stable node.

3. When $c_1 \ and \ c_2$ are both positive, real and distinct, the equilibrium point (λ_p, γ_p) is an unstable node.

Remark 4: The first two cases above confirm the non-existence of limit cycle and the third the existence of a unique stable limit cycle.

MAIN RESULTS

Two isoclines of particular interest in this section are the prey and predator isoclines as defined in Freedman (1980). By his definition the right hand side of the first equation in system (1) is set equal to zero and the prey isocline is the curve obtained for a fixed p:

$$y = rx\left(1 - \frac{x}{k}\right) / \frac{x^p}{a + x^p} \qquad \textbf{(7)}$$

By setting $y' = 0$ in the second equation of system (1) the predator isocline is defined by the line $x = \lambda_p$ for a fixed p.

Consider another system different from (1)

$$x' = xg(x) - yp(x)$$

$$y' = y(-v + q(x)) \qquad \textbf{(8)}$$

Freedman (1980) derived the following criterion, using the prey isocline for system (8).

$$\frac{xg(x)}{p(x)}\begin{cases} \textit{decreasing at } x^* \textit{ implies } x^* \textit{ is stable} \\ \textit{increasing at } x^* \textit{ implies } x^* \textit{ is unstable}\end{cases}$$

Here, x^* is the value of the potential interior equilibrium point. Our first result is a derivation of this criterion for system (1). The equilibrium point using the predator isocline for it is $x = \lambda_p$. Comparing systems (1) and (8) leads to the following theorem.

Theorem D

If the prey isoclines:

$$y = rx\left(1 - \frac{x}{k}\right) / \frac{x^p}{a + x^p} \qquad \textbf{(9)}$$

is decreasing at λ_p, this implies that λ_p is stable and if it is increasing at λ_p, this implies that λ_p is unstable.

Before the proof of this theorem, we state and prove the following lemmas.

Lemma 1

Suppose that the prey isoclines of system (1) is defined by Equation (7) then:

$$\left.\frac{1}{y}\frac{dy}{dx}\right|_{x=\lambda_p} = \frac{\tau}{r\lambda_p\left(1 - \frac{\lambda_p}{k}\right)}. \qquad \textbf{(10)}$$

Proof: Rewrite Equation (9) as $y = r\left(x - \frac{x^2}{k}\right)(ax^{-p} + 1)$ and differentiate with respect to x to get:

$$\frac{dy}{dx} = r\left(1 - \frac{2x}{k}\right)(ax^{-p} + 1) - rapx^{-p-1}\left(x - \frac{x^2}{k}\right)$$

and

$$\frac{1}{y}\frac{dy}{dx} = \frac{\left(1 - \frac{2x}{k}\right)(ax^{-p} + 1) - apx^{-p-1}\left(x - \frac{x^2}{k}\right)}{\left(x - \frac{x^2}{k}\right)(ax^{-p} + 1)} \qquad \textbf{(11)}$$

On evaluation of $\frac{1}{y}\frac{dy}{dx}$ at $x = \lambda_p$ and simplification of right hand side of equation (11), we obtain the desired result, which is (10), where $\tau = \frac{r}{\mu k}\left[\left((1-p)\mu + pD\right)k + \left((p-2)\mu - pD\right)\lambda_p\right]$.

Next, we state and prove the second lemma, which gives another representation of $\frac{1}{y}\frac{dy}{dx}$ at $x = \lambda_p$.

Lemma 2

Let the prey isocline of system (1) be defined by Equation (7). Then:

$$\frac{\tau}{r\lambda_p\left(1 - \frac{\lambda_p}{k}\right)} = \left.\frac{d}{dx} ln\left[\frac{rx\left(1 - \frac{x}{k}\right)}{\frac{x^p}{a + x^p}}\right]\right|_{x=\lambda_p} \qquad \textbf{(12)}$$

Proof: Rewrite Equation (9) as:

$$lny = ln\left[\frac{rx\left(1 - \frac{x}{k}\right)}{\frac{x^p}{a + x^p}}\right] \qquad \textbf{(13)}$$

Differentiating Equation (13) with respect to x leads to:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx} ln\left[\frac{rx\left(1 - \frac{x}{k}\right)}{\frac{x^p}{a + x^p}}\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{\frac{1}{k}}{\left(1 - \frac{x}{k}\right)} - \frac{apx^{-p-1}}{(ax^{-p} + 1)},$$

Using Lemma 1, the desired result (12) follows.

Proof of Theorem D: Since τ is given by $\tau = \frac{r}{\mu k}\left[\left((1-p)\mu + pD\right)k + \left((p-2)\mu - pD\right)\lambda_p\right]$

and both λ_p and $\left(1-\frac{\lambda_p}{k}\right)$ are positive, $\frac{\tau}{r\lambda_p\left(1-\frac{\lambda_p}{k}\right)}$ has the same sign as τ.

From Lemma 2, $\frac{\tau}{r\lambda_p\left(1-\frac{\lambda_p}{k}\right)} = \frac{d}{dx} ln\left[\frac{rx\left(1-\frac{x}{k}\right)}{\frac{x^p}{a+x^p}}\right]\Bigg|_{x=\lambda_p}$ and from "Non-Existence of Limit Cycles" it is seen that $\tau > 0$ implies (λ_p, γ_p) is unstable and $\tau < 0$ implies (λ_p, γ_p) is stable. The conclusion is that:

$$\frac{d}{dx} ln\left[\frac{rx\left(1-\frac{x}{k}\right)}{\frac{x^p}{a+x^p}}\right]\Bigg|_{x=\lambda_p} < 0 \qquad \textbf{(14a)}$$

implies that (λ_p, γ_p) is stable and

$$\frac{d}{dx} ln\left[\frac{rx\left(1-\frac{x}{k}\right)}{\frac{x^p}{a+x^p}}\right]\Bigg|_{x=\lambda_p} > 0 \qquad \textbf{(14b)}$$

implies that (λ_p, γ_p) is unstable.

But the derivative of y is positive or negative if and only if y itself is increasing or decreasing, respectively, and log is an increasing function means that inequalities (14a) and (14b) are equivalent to $rx\left(1-\frac{x}{k}\right)(ax^{-p}+1)$ decreasing at λ_p implies (λ_p, γ_p) is stable, and $rx\left(1-\frac{x}{k}\right)(ax^{-p}+1)$ is increasing at λ_p implies (λ_p, γ_p) is unstable (Freedman, 1980). This proves the theorem.

GRAPHICAL INTERPRETATION OF CRITERION

Here, we give a graphical representation of this criterion with interpretation. The predator isocline, $x = \lambda_p = \left(\frac{aD}{\mu-D}\right)^{\frac{1}{p}}$, is not necessarily a vertical line and the sign of the real parts of the eigenvalues of J is not necessarily given by τ. So we consider fixing $p = 1\ or\ p = 2$ to give an interpretation of the criterion. In this case, $x = \lambda_1\ or\ \lambda_2$ is a vertical line.

For $p = 1$, $x = \lambda_1 = \frac{aD}{\mu-D}$ and since $y = rx\left(1-\frac{x}{k}\right)(ax^{-1}+1)$, the prey isocline intersects the predator isocline at $\left(\lambda_1, r\left(\lambda_1-\frac{\lambda_1^2}{k}\right)\left(\frac{a}{\lambda_1}+1\right)\right)$, or at $\left(\lambda_1, r\left(1-\frac{\lambda_1}{k}\right)(a+\lambda_1)\right)$ and it goes through the points $(k,0)\ and\ (0,ar)$.

Since $\frac{dy}{dx} = r\left(1-\frac{2x}{k}\right)(ax^{-1}+1) - r(ax^{-2})\left(x-\frac{x^2}{k}\right)$, set $\frac{dy}{dx} = 0$ so that $r\left(1-\frac{2x}{k}\right)(ax^{-p}+1) - ax^{-2}\left(x-\frac{x^2}{k}\right) = 0$, and this reduces to $-2x^2+kx-ax = 0$. This quadratic equation has roots $x = 0\ and\ x = \frac{k-a}{2}$.

It follows from the second root that $k > a$. Differentiating once more gives $\frac{d^2y}{dx^2} = -\frac{2r}{k} < 0$. This shows that there is a local maximum at $\left(\frac{k-a}{2}, \frac{r(a+k)^2}{4k}\right)$. This local maximum can be right or left of the point (λ_1, γ_1) of intersection of prey and predator isoclines according as the equilibrium point (λ_1, γ_1) is asymptotically stable or unstable.

We present the following examples to illustrate the criterion for the given values of the parameters $r, k, a, D, and\ \mu$.

Example 1:

$$x' = x\left(1-\frac{x}{1.5}\right) - \frac{x}{1+x}y$$

$$y' = y\left(\frac{1.5x}{1+x} - 0.5\right)$$

The point A (λ_1, γ_1) is stable and at A, the prey isocline is decreasing. Assumption 1.2 holds but there is no limit cycle since Theorem A is not satisfied for the point of interaction is to the right of the local maximum.

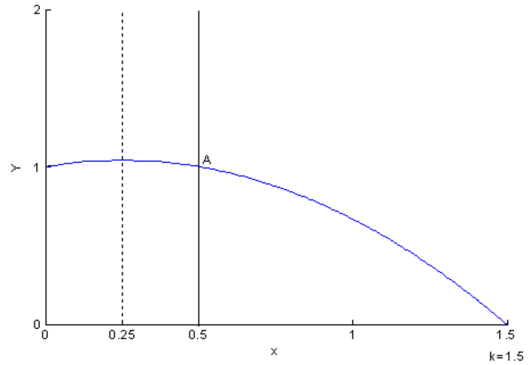

Figure 1: Non-Existence of Limit Cycle since A is to the right of B Local Maximum.

Example 2:

$$x' = x\left(1-\frac{x}{4}\right) - \frac{x}{1+x}y$$

$$y' = y\left(\frac{3x}{1+x} - 1.5\right)$$

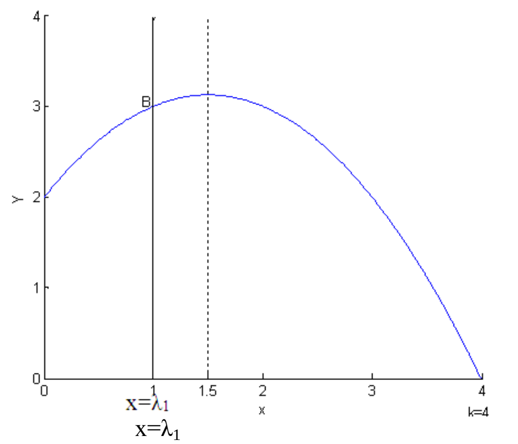

Figure 2: Existence of Limit Cycle since A is to the left of the Local Maximum.

The point A is unstable and at A $\tau > 0$ and $\lambda_1 < k$. By our criterion $(D+\mu)\lambda_1 \leq Dk$. This is essentially the substance of Theorem A; we conclude that there is a unique stable limit cycle.

For $p = 2$, an analysis similar to that for $p = 1$ gives the two isoclines of system(1) intersecting at (λ_2, γ_2), where $\lambda_2 = \left(\frac{aD}{\mu - D}\right)^{\frac{1}{2}}$,and $\gamma_2 = r\left(1-\frac{\lambda_2}{k}\right)\left(a\lambda_2^{-1}+\lambda_2\right)$ and the prey isocline goes through the point $(k, 0)$ and (λ_2, γ_2) but it does not intersect the y $axis$.

Now, we consider two examples with $p = 2$ and parameters the same as those of Examples 1 and 2.

Example 3:

$$x' = x\left(1-\frac{x}{1.5}\right) - \frac{x^2}{1+x^2}y$$

$$y' = y\left(\frac{1.5x^2}{1+x^2} - 0.5\right)$$

The values of λ_2 and γ_2 are $\sqrt{2}$ and $\left(\frac{3}{\sqrt{2}} - \frac{2}{3}\right)$, respectively. $\tau = \frac{1}{(1.5)^2}\left[-\left(\frac{3}{4}+\sqrt{2}\right)\right] < 0$ and $\frac{\tau}{\lambda_2\left(1-\frac{\lambda_2}{k}\right)} = \frac{1}{(1.5)^2}\frac{\left[-\left(\frac{3}{4}+\sqrt{2}\right)\right]}{\sqrt{2}\left(1-\frac{\sqrt{2}}{1.5}\right)} < 0$. So $\tau < 0$ implies that $(\lambda_2, \gamma_2) = \left(\sqrt{2}, \frac{3}{\sqrt{2}} - \frac{2}{3}\right)$ is stable and the point $\lambda_2 = \sqrt{2}$ is on the right of the local maximum. By Theorem D, $y = (1+x^2)\left(1-\frac{x}{1.5}\right)$ is decreasing at λ_2. We conclude that there is no limit cycle.

Example 4:

$$x' = x\left(1-\frac{x}{4}\right) - \frac{x^2}{1+x^2}y$$

$$y' = y\left(\frac{3x^2}{1+x^2} - 1.5\right)$$

The values of λ_2 and γ_2 are 1 and $\frac{3}{2}$, respectively. $\tau = \frac{1}{4} > 0$ and $\lambda_2\left(1-\frac{\lambda_2}{k}\right) = \frac{3}{4} > 0$. By Theorem D, the prey isocline is increasing at $\left(1, \frac{3}{2}\right)$ implies that $\left(1, \frac{3}{2}\right)$ is unstable, and hence there is a unique stable limit cycle, since the local maximum is on the right of $\lambda_2 = 1$.

CONCLUSION

In Examples 3 and 4 the criterion tell us that the local maximum is on the left of $\lambda_2 = \sqrt{2}$ and on the right of $\lambda_2 = 1$ respectively. For these two examples, the position of the local maxima are not given as precise as for the case $p = 1$. The reader interested in this may will have to solve a cubic equation a rising from differentiating the prey isoclines in each and from the root find the position of local maxima if they exists.

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