Reformulation of Runge-Kutta Method into Linear Multistep Method for Error and Convergence Analysis.

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ABSTRACT

Error analysis in Runge-Kutta Method (RKM) is considerably more difficult than in the case of Linear Multistep Method (LMM) due to loss of linearity in the method, also the fact that the general RKM makes no mention of the function f(x,y), which defines the differential equation, makes it impossible to define the order of the method independently of the differential equation. In this paper, we reformulate RKM into a linear multistep method which goes some way toward overcoming these deficiencies the method. Through the idea in this paper, we shall be able to determine the orders, error constant, zero-stable, consistent and convergence of the RKM independently of the differential equation via a similar process like linear multistep method.

(Keywords: Runge-Kutta method, RKM, linear multistep method, LMM, differential equation)

INTRODUCTION

The initial value problem (IVP) for a system of first order Ordinary Differential Equations (ODEs) is defined by:

$$y' = f(x, y) \qquad y(x_o) = y \qquad x \in [a, b] \qquad \textbf{(1)}$$

The general s-stage Runge-Kutta method is defined by:

$$y_{n+1} = y_n + h\phi(x_n, y_n, h),$$

$$\phi(x_n, y_n, h) = \sum_{i=1}^{s} b_i k_i \qquad \textbf{(2)}$$

$$K_i = f(x_n + c_j h, y_n + h\sum_{j=1}^{s} \bar{a}_{ij} k_j) \qquad \textbf{(3)}$$

Where for i = 1, 2 - - - - s, the real parameters c_j, k_i, a_{ij} define the method and s is the stage number.

Such formula can be represented conveniently by Butcher array:

$$\begin{array}{c|ccccc} c_1 & a_{11} & a_{12} & a_{13} & \dots & a_{1s} \\ c_2 & a_{21} & a_{22} & a_{23} & \dots & a_{2s} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ c_s & a_{s1} & a_{s2} & a_{s3} & \dots & a_{ss} \\ \hline & b_1 & b_2 & b_3 & \dots & b_s \end{array} \qquad \textbf{(4)}$$

Or simply as:

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \qquad \textbf{(5)}$$

According to Kulikov (2003),if matrix A is strictly lower triangular (i.e., the internal stages can be calculated without depending on later stages),then the method is called an explicit method otherwise the internal stages depend not only on the previous stages but also on the current stage and later stages, this method is called an implicit method. The importance of the implicit methods is due to its high orders of accuracy which is superior to the explicit methods. This makes it more suitable for solving stiff problems.

In order to derive a fifth order Runge-Kutta method, seventeen equations have to be satisfied [5]. The equations associated with the order of the method are given in Table 1.

Table 1: Equations of Order Conditions for Runge- Kutta Methods of Order 5.

order	*Elementary* Weight
1	$\sum \mathrm{b}_i = 1$
2	$\sum \mathrm{b}_i c_i = \frac{1}{2}$
3	$\sum \mathrm{b}_i c_i^2 = \frac{1}{3}$
3	$\sum \mathrm{b}_i a_{ij} c_i = \frac{1}{6}$
4	$\sum \mathrm{b}_i c_i^3 = \frac{1}{4}$
4	$\sum \mathrm{b}_i c_i a_{ij} c_i = \frac{1}{8}$
4	$\sum \mathrm{b}_i a_{ij} c_i^2 = \frac{1}{12}$
4	$\sum \mathrm{b}_i a_{ij} a_{jk} c_k = \frac{1}{24}$
5	$\sum \mathrm{b}_i c_i^4 = \frac{1}{5}$
5	$\sum \mathrm{b}_i c_i^2 a_{ij} c_i = \frac{1}{10}$
5	$\sum \mathrm{b}_i a_{ij} c_j a_{jk} c_k = \frac{1}{20}$
5	$\sum \mathrm{b}_i c_i a_{ij} c_i^2 = \frac{1}{15}$
5	$\sum \mathrm{b}_i a_{ij} c_i^3 = \frac{1}{20}$
5	$\sum \mathrm{b}_i c_i a_{ij} a_{jk} c_k = \frac{1}{30}$
5	$\sum \mathrm{b}_i a_{ij} c_j a_{jk} c_k = \frac{1}{40}$
5	$\sum \mathrm{b}_i a_{ij} a_{jk} c_k^2 = \frac{1}{60}$
5	$\sum \mathrm{b}_i a_{ij} a_{jk} a_{kl} c_l = \frac{1}{120}$

On the other hand for the general Linear Multstep Method (LMM) is defined by:

$$\sum_{\mathrm{j}=0}^{\mathrm{k}} \alpha_{\mathrm{j}} \mathrm{y}_{\mathrm{n+j}} = h \sum_{\mathrm{j}=0}^{\mathrm{k}} \beta_{\mathrm{j}} \mathrm{f}_{\mathrm{n+j}} \qquad \textbf{(6)}$$

Where are α_{j} and β_{j} constant is number of step and h is the step size.

Definition 1

The LMM (6) is of order P if $c_0 = c_1 = c_2 = \ldots = c_p = 0$ but $c_{p+1} \neq 0$ and c_{p+1} is called the error constant, where

$c_0 = \alpha_0 + \alpha_1 + \ldots + \alpha_k$,

$c_1 = (\alpha_1 + 2\alpha_2 + \ldots + k\alpha_k) - (\beta_0 + \beta_1 + \ldots + \beta_k)$

$$c_q = \frac{1}{q}!(\alpha_1 + 2^q \alpha_2 + \ldots + k^q \alpha_k) - \frac{1}{q-2}!(\beta_1 + 2^{q-2}\beta_2 + \ldots + k^{q-k}\beta_k) \qquad \textbf{(7)}$$

Definition 2

The LMM (6) is said to be consistent if it has order p$\geq$1, from (7), and

$$\sum_{j=0}^{k} \alpha_j = 0,$$

$$\sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j = 0 \qquad \textbf{(8)}$$

By introducing the first and second characteristics polynomials of the LMM (1.6) defined as:

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$$

$$\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j \qquad \textbf{(9)}$$

Then from (8) the LMM (6) is consistent if $\rho(1) = 0, \quad \rho'(1) = \sigma(1)$

Definition 3

The LMM (6) is said to be zero-stable if no root of the first characteristics polynomial $\rho(\xi)$ has modulus greater than one, and if every root of the modulus one has multiplicity not greater than two[3].

Theorem 1 (Convergence)

The necessary and sufficient condition for a LMM to be convergence are that it must be consistent and zero–stable [Dahlquist.(1956)] with the addition of the hypothesis

Lim $(\eta_{\mu}^{(h)} - \eta_{0}^{(h)})$/h= $\hat{\eta}$ **(10)**

The paper presents how the reformulation was constructed, where the error and convergence analysis were discussed, and the conclusions which are drawn 3.

THE REFORMULATION

In this section for $c_1, c_2, \ldots, c_s$ and $k_1, k_2, \ldots, k_s$ in (3), we shall let $k_i = f_{c_i}$ implies $k_1 = f_{c_1}$,

$$k_2 = f_{c_2}, k_3 = f_{c_3}, \ldots, k_s = f_{c_s} \quad \textbf{(11)}$$

(i) Consider the well known third-order Runge-Kutta method

$$y_{n+1} = y_n + \frac{h}{4}(k_1 + 3k_3), \quad \textbf{(12)}$$

$$K_1 = f(x_n, y_n) \quad \textbf{(13)}$$

$$K_2 = f(x_n + \frac{1}{3}h, y_n + \frac{1}{3}hk_1)$$

$$K_3 = f(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_2)$$

From equation (13) $c_1 = 0,\ c_2 = \frac{1}{3}$ and $c_3 = \frac{2}{3}$

Appling Equation (1) $k_1 = f_0$ $k_2 = f_{\frac{1}{3}}$ $k_3 = f_{\frac{2}{3}}$ thus Equation (12) becomes:

$$y_{n+1} = y_n + \frac{h}{4}(f_0 + 3f_{\frac{2}{3}}) \quad \textbf{(14)}$$

(ii) Consider the well known fourth-order Runge-Kutta method:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad \textbf{(15)}$$

$$K_1 = f(x_n, y_n) \quad \textbf{(16)}$$

$$K_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1)$$

$$K_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$$

$$K_4 = f(x_n + h, y_n + hk_3)$$

From Equation (16) $c_1 = 0,\ c_2 = c_3 = \frac{1}{2}$ and $c_4 = 1$.

Appling Equation (11) $k_1 = f_0$, $k_2 = k_3 = f_{\frac{1}{2}}$ and $k_4 = f_1$ thus Equation (15) become

$$y_{n+1} = y_n + \frac{h}{6}(f_0 + 4f_{\frac{1}{2}} + f_1) \quad \textbf{(17)}$$

(iii) Consider the well known fifth-order Runge-Kutta method:

$$y_{n+1} = y_n + \frac{h}{192}(23k_1 + 125k_3 - 81k_5 + 125k_6) \quad \textbf{(18)}$$

$$K_1 = f(x_n, y_n) \quad \textbf{(19)}$$

$$K_2 = f(x_n + \frac{1}{3}h, y_n + \frac{1}{3}hk_1)$$

$$K_3 = f(x_n + \frac{2}{5}h, y_n + \frac{1}{25}h(4k_1 + 6k_2)$$

$$K_4 = f(x_n + h, y_n + \frac{1}{4}h(k_1 - 12k_2 + 15k_3)$$

$$K_5 = f(x_n + \frac{2}{3}h, y_n + \frac{1}{81}h(6k_1 + 90k_2 - 50k_3 + 8k_4)$$

$$K_6 = f(x_n + \frac{4}{5}h, y_n + \frac{1}{75}h(6k_1 + 36k_2 + 10k_3 + 8k_4)$$

From Equation (19):

$$c_1 = 0, \; c_2 = \frac{1}{3}, c_3 = \frac{2}{5}, c_4 = 1, c_5 = \frac{2}{3} \text{ and } c_6 = \frac{4}{5}$$

Appling equation (11) $k_1 = f_0$, $k_2 = f_{\frac{1}{3}}$, $k_3 = f_{\frac{2}{5}}$, $k_4 = f_1$, $k_5 = f_{\frac{2}{3}}$ and $k_6 = f_{\frac{4}{5}}$ thus Equation (18) becomes:

$$y_{n+1} = y_n + \frac{h}{192}(23f_0 + 125f_{\frac{2}{5}} - 81f_{\frac{2}{3}} + 125f_{\frac{4}{5}}) \quad \textbf{(20)}$$

(iv) Consider the sixth-order implicit Runge-Kutta method [2]:

$$y_{n+1} = y_n + \frac{h}{90}(-9k_1 + 32k_2 + 44k_3 + 32k_4 - 9k_5) \quad \textbf{(21)}$$

$$K_1 = f(x_n, y_n) \quad \textbf{(22)}$$

$$K_2 = f(x_n + (\frac{1}{2} - \frac{\sqrt{3}}{4})h, y_n$$
$$+ h(a_{11}k_1 + a_{12}k_2 + a_{13}k_3 + a_{14}k_4 + a_{15}k_5))$$

$$K_3 = f(x_n + \frac{1}{2}h, y_n$$
$$+ h(a_{11}k_1 + a_{12}k_2 + a_{13}k_3 + a_{14}k_4 + a_{15}k_5))$$

$$K_4 = f(x_n + (\frac{1}{2} + \frac{\sqrt{3}}{4})h, y_n$$
$$+ h(a_{11}k_1 + a_{12}k_2 + a_{13}k_3 + a_{14}k_4 + a_{15}k_5))$$

$$K_5 = f(x_n + h, y_n +$$
$$h(a_{11}k_1 + a_{12}k_2 + a_{13}k_3 + a_{14}k_4 + a_{15}k_5))$$

From Equation (22)

$$c_1 = 0, \; c_2 = \frac{1}{2} - \frac{\sqrt{3}}{4}, c_3 = \frac{1}{2}, c_4 = \frac{1}{2} + \frac{\sqrt{3}}{4} \text{ and } c_5 = 1$$

Appling Equation (11) $k_1 = f_0$, $k_2 = f_{\frac{1}{2}-\frac{\sqrt{3}}{4}}$, $k_3 = f_{\frac{1}{2}}$, $k_4 = f_{\frac{1}{2}+\frac{\sqrt{3}}{4}}$ and $k_5 = f_1$ thus Equation (21) becomes:

$$y_{n+1} = y_n + \frac{h}{90}(-9f_0 + 32f_{\frac{1}{2}-\frac{\sqrt{3}}{4}} + 44f_{\frac{1}{2}} + 32f_{\frac{1}{2}+\frac{\sqrt{3}}{4}} - 9f_1) \quad \textbf{(23)}$$

ERROR AND CONVERGENCE ANALYSIS OF THE NEW METHODS

In this section we shall be investigating the order, error constant and zero stability of each of the newly reformulated methods.

(i) In (14) $\alpha_1 = 1, \quad \alpha_0 = -1 \quad \beta_0 = \frac{1}{4}$ and $\beta_{\frac{2}{3}} = \frac{3}{4}$

Using (6), (7) and Definition (1) $c_0 = c_1 = c_2 = c_3 = 0$ but $c_{p+1} = c_4 = \frac{1}{216}$

The order=3 and the error constant is $\frac{1}{216}$

By Definition (2) the method is consistent. For zero stability:

First characteristics polynomial $\rho(\xi) = \varepsilon - 1 = 0, \quad \text{Root} \quad \varepsilon = 1, \ |\varepsilon| = 1$

By Definition (3) the method is zero-stable. Hence, the method is convergent by Theorem (1).

(ii) In (17) $\alpha_1 = 1, \quad \alpha_0 = -1 \quad \beta_0 = \frac{1}{6}, \beta_{\frac{1}{2}} = \frac{4}{6}$ and $\beta_1 = \frac{1}{6}$

Using (6), (7), and Definition (1) $c_0 = c_1 = c_2 = c_3 = c_4 = 0$, but $c_{p+1} = c_5 = -\frac{1}{2880}$.

The order=4 and the error constant is $-\frac{1}{2880}$

By Definition (2), the method is consistent. For zero stability:

First characteristics polynomial $\rho(\xi) = \varepsilon - 1 = 0, \quad \text{Root} \quad \varepsilon = 1, \ |\varepsilon| = 1$

By Definition (3) the method is zero-stable. Hence, the method is convergent by Theorem (1).

Using the same procedure of obtaining the order and the error constant as above, Methods (14), (17), (20) and (23) gives the following results, in tabular form:

Table 2: Tabular Results.

Method	Order	Error Constant
$y_{n+1} = y_n + \frac{h}{4}(f_0 + 3f_{\frac{2}{3}})$	3	4.630×10^{-3}
$y_{n+1} = y_n + \frac{h}{6}(f_0 + 4f_{\frac{1}{2}} + f_1)$	4	3.472×10^{-4}
$y_{n+1} = y_n + \frac{h}{192}(23f_0 + 125f_{\frac{2}{5}} - 81f_{\frac{2}{3}} + 125f_{\frac{4}{5}})$	5	1.852×10^{-5}
$y_{n+1} = y_n + \frac{h}{90}(-9f_0 + 32f_{\frac{1}{2}-\frac{\sqrt{3}}{4}} + 44f_{\frac{1}{2}} + 32f_{\frac{1}{2}+\frac{\sqrt{3}}{4}} - 9f_1)$	6	9.3005×10^{-7}

The methods are all consistent and zero-stable; hence they are all convergent.

CONCLUSION

The advantage this approach is that it speed up computation and reduces computational efforts in determines the order and convergence of RKM. The approach also allows arbitrary higher order to be formulated (order greater than 5 without using Table 1). Finally because of the success of the reformulation, research later will be focused on reformulating RKM (including the stages) into LMM for the solution of IVPs.

REFERENCES

1. Kulikov, G. Yu. 2003. "Symmetric Runge Kutta Method and their Stability". *Russ J. Numeric Analyze and Maths. Modeling.* 18(1):13-41.

2. Yakub, D.G., Markus, S., Adamu, M.Y., and Buba S.S. 2007. "A New Formulation for Symmetric Runge Kutta Method Initial Value Problems of Ordinary Differential Equations". *Abacus*. 34(2A) Math Series:193-205.

3. Lambert, J.D. 1973. *Computational Methods in Ordinary Differential Equations*. John Willey and Sons: New York, NY.

4. Butcher, J.C. 2003. *Numerical methods for Ordinary Differential Equations*. John Willey and Sons: New York, NY.

5. Ababneh, O.Y., R. Ahmad, and E.S. Ismail. 2009. "Design of New Diagonally Implicit Runge–Kutta Methods for Stiff Problems". *Applied Mathematical Sciences*. 3(45):2241 – 2253.

6. Dahlquist, G. 1956. "Convergence and Stability in the Numerical Integration of Ordinary Differential Equations". BII.18:133-136.

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