On the Convergence, Consistence, and Stability of a New One-Step Method Based on the Combination of Two Interpolating Functions.

E.A. Ibijola[1]; O.O. Bamisile[1]; and J. Sunday[2]

[1]Department of Mathematical Sciences, University of Ado, Ekiti, Nigeria.
[2]Department of Mathematical Sciences, Adamawa State University, Mubi, Nigeria.
Awka, Anambra State, Nigeria.

E-mail: bode_bamisile@yahoo.com*

ABSTRACT

This paper presents convergence, consistence, and stability analysis of a One-Step Finite-Difference Scheme developed by Ibijola and Bamisile (2009). The New Scheme was applied to some initial value problems and has proven to be efficient and computationally reliable.

(Keywords: convergence, consistency, stability, initial value problems)

INTRODUCTION

It is a well known fact that a given linear differential equation does not have a complete solution that can be expressed in terms of a finite number of elementary functions. It is also a known fact that one of the ways of solving such problems is to seek approximate solutions by means of various perturbation methods (Rose 1964, Humi and Miller 1989). It must be stated here that the above procedure will only hold for limited ranges of system parameters and the independent variable (Mickens 1994). As reported in Mickens (1994) for arbitrary values of the system parameters a particular time, only numerical integration techniques can provide accurate numerical solutions to the original differential equations. That is why Ibijola and Bamisile (2009) developed a new One-Step Standard Finite Difference Scheme capable of solving of the form:

$$y' = f(x,y),\ y(a) = \eta,\ x \in [a,b] \quad \textbf{(1)}$$

Let us consider the theorem 1 below which guarantees the existence of a unique solution of any problem of the form (1).

Theorem 1 (Lambert 1973, Fatunla 1976, 1988)

Let $f(x,y)$ be defined and continuous for all points (x,y) in the region D defined by $\{(x,y): a \le x \le b, -\infty < y < \infty\}$ where $a\ and\ b$ finite, and let there exist a constant L such that for every x, y, y^* such that $(x,y)\ and\ (x,y^*)$ are both in D:

$$\left| f(x,y) - f(x,y^*) \right| \le L \left| y - y* \right| \quad \textbf{(2)}$$

Then, if η is any given number, there exist a unique solution $y(x)$ of the initial value problem (1). The inequality (2) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

Definition 1 (Henrici 1962)

Any method for solving differential equations in which the approximation y_{n+1} to the solution at the point x_{n+1} can be calculated if only $x_n, y_n,\ and\ h$ are known, is called a ONE-STEP METHOD. It is a common fact to write the functional dependence y_{n+1} on the quantities $x_n, y_n,\ and\ h$ in the form:

$$y_{n+1} = y_n + h\phi(x_n, y_n; h) \quad \textbf{(3)}$$

where $\phi(x_n, y_n; h)$ is called the increment function.

Definition 2 (Convergence)

A numerical scheme is said to be convergent if for all initial value problem (1) satisfying the hypothesis of Lipshitz condition, $\max_{0\le n\le N}\|y(x_n)-y_n\| \to 0$ as $h \to 0$ if for any arbitrary point $x \in [a,b]$, the global error fulfils the following relationship; $\lim_{h\to 0} Max E_n \to 0$, provided x is always a mesh point.

Definition 3 (Consistence)

A numerical scheme with an increment function $\phi(x_n, y_n; h)$ is said to be consistent with the initial value problem (1), if $\phi(x_n, y_n; 0) = f(x_n, y_n)$. Note that if the scheme developed is consistent with the initial value problem (1), then, $h = 0$. The concept of consistency of one-step method is very crucial in the sense that it controls the magnitude of the local truncation error.

Stability (Henrici 1962)

A numerical solution of the class of system (1) is said to be stable if the difference between the numerical solution and the theoretical solution can be made as small as possible, i.e. if there exists two positive numbers e_0 and K such that the following holds: $\|y_n - y(x_n)\| \le K\|e_0\|$

Theorem 2

Let $y_n = y(x_n)$ and $l_n = l(x_n)$ denote two different numerical solutions of differential equations under consideration with the initial conditions specified as $y(x_0) = \eta$ and $l(x_0) = \eta^*$ respectively, such that $|\eta - \eta^*| < \varepsilon$, $\varepsilon > 0$. If the two numerical estimates we generated are:

$$y_{n+1} = y_n + h\phi(x_n, y_n; h) \quad \textbf{(4)}$$

$$l_{n+1} = l_n + h\phi(x_n, l_n; h) \quad \textbf{(5)}$$

The condition that:

$$|y_{n+1} - l_{n+1}| \le K|\eta - \eta^*| \quad \textbf{(6)}$$

is the necessary and sufficient condition that our new method be stable and convergent.

We shall now consider the derivation, convergence, consistency, stability and applications of the New One-Step Standard Finite Difference Scheme.

Derivation of the New One-Step Scheme (Ibijola and Bode 2009)

Let us assume that the theoretical solution $y(x)$ to the initial value problem (1) can be locally represented in the interval $[x_n, x_{n+1}]$, $n \ge 0$ by the non-polynomial interpolating function:

$$F(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4\left(e^{\lambda x_n}\right) + bCosx_n$$

and assume that at the points $x = x_n$ and $x = x_{n+1}$, we have:

$$F(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4\left(e^{\lambda x_n}\right) + bCosx_n \quad \textbf{(7)}$$

And

$$F(x_{n+1}) = a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4\left(e^{\lambda x_n+1}\right) + bCosx_{n+1} \quad \textbf{(8)}$$

If we also assume that $F(x_n)$ and $F(x_{n+1})$ coincide with y_n and y_{n+1} respectively and that $F^{(i)}$ denote the ith total derivatives of $f(x, y)$ with respect to x_n in the same manner. We shall adopt the series expansion for the exponential functions, which we have as:

$$e^{\lambda x_n} = \sum_{r=0}^{\infty} \frac{\lambda^r x_n^r}{r!} = 1 + \lambda x_n \frac{(\lambda x_n)^2}{2!} + \ldots \quad \textbf{(9)}$$

$$(e^{\lambda h} - h) = \sum_{r=0}^{\infty} \frac{\lambda^r h^r}{r!} = 1 + \lambda h + \frac{(\lambda h)^2}{2!} + \ldots \quad \textbf{(10)}$$

We denote the ith total derivative of $f(x, y)$ with respect to x with $f^{(ii)}$ such that:

$$F(x_n) = y_n = a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4\left(1 + \lambda x_n + \frac{(\lambda x)^2}{2!} + \frac{(\lambda x)^3}{3!} + \frac{(\lambda x)^4}{4!} + \ldots\right) + bCosx_n \quad \textbf{(11)}$$

$$F^{(1)}(x_n) = f_n = a_1 + 2a_2 x_n + 3a_3 x_n^2 + a_4\left(\lambda + \lambda^2 x + \frac{\lambda^3 x^2}{2!} + \frac{\lambda^4 x^3}{3!} + \ldots\right) - bSinx_n \quad \textbf{(12)}$$

$$F^{(2)}(x_n) = f_n^{(1)} = 2a_2 + 6a_3 x_n + a_4\left(\lambda^2 + \lambda^3 x_n + \frac{\lambda^4 x^2}{2!} + \ldots\right) - bCosx_n \quad \textbf{(13)}$$

$$F^{(3)}(x_n) = f_n^2 = 6a_3 + a_4\left(\lambda^3 + \lambda^4 x_n + \ldots\right) + bSinx_n \quad \textbf{(14)}$$

$$F^{(4)}(x_n) = f_n^3 = a_4 \lambda^4 + bCosx_n \quad \textbf{(15)}$$

$$F^{(5)}(x_n) = f_n^4 = -bSinx_n \quad \textbf{(16)}$$

$$b = \frac{-f_n^4}{Sinx_n} \quad \textbf{(17)}$$

Putting (17) into (15) we have: $a_4 = \frac{1}{\lambda^4}\left(f_n^3 + f_n^4 ATAN(x_n)\right)$ **(18)**

Putting Equations (17) and (18) into (14) gives:

$$a_3 = \frac{1}{6}\left\{f_n^2 - \left(f_n^3 + f_n^4 ATAN(x_n)\right)\left(\tfrac{1}{\lambda} + x_n\right) + f_n^4\right\} \quad \textbf{(19)}$$

Putting Equations (17), (18) and (19) into Equation (13) also gives:

$$a_2 = \frac{1}{2}\left\{\begin{matrix} f_n^1 - \left(f_n^2 - (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda} + x_n) + f_n^4\right)x_n - \\ (f_n^3 + f_n^4 ATAN(x_n)(\tfrac{1}{\lambda^2} + \tfrac{x_n}{\lambda} + \tfrac{x_n^2}{2}) - f_n^4 ATAN(x_n) \end{matrix}\right\} \quad \textbf{(20)}$$

Substituting Equations (17), (18), (19) and (20) into (12) gives:

$$a_1 = \left\{\begin{matrix} f_n - \left[\begin{matrix} f_n^1 - \left(f_n^2 - (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda} + x_n) + f_n^4\right)x_n - \\ (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda^2} + \tfrac{x_n}{\lambda} + \tfrac{x_n^2}{2}) - f_n^4 ATAN(x_n) \end{matrix}\right] x_n - \\ \frac{1}{2}\left(f_n^2 - (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda} + x_n) + f_n^4\right)x_n^2 - \\ (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda^3} + \tfrac{x_n}{\lambda^2} + \tfrac{x_n^2}{2\lambda} + \tfrac{x_n^3}{3}) - f_n^4 \end{matrix}\right\} \quad \textbf{(21)}$$

From our assumption that $F(x_n) = y_n$ and $F(x_{n+1}) = y_{n+1}$, if we proceed to substitute Equations (17), (18), (19), (20) and (21) in Equations (7) and (8) respectively we have the following:

$$
\begin{aligned}
& y_{n+1} - y_n = a_1(x_{n+1} - x_n) + a_2(x_{n+1}^2 - x_n^2) + \\
& a_3(x_{n+3}^3 - x_n^3) + a_4(e^{\lambda x_{n+1}} - e^{\lambda x_n}) + b(\cos x_{n+1} - x_n)
\end{aligned}
\qquad \textbf{(22)}
$$

Using $x_{n+1} - x_n = h$, $x_{n+1}^2 - x_n^2 = 2x_n h + h^2$ and $x_{n+1}^3 - x_n^3 = 3x_n^2 h + 3x_n h^2 + h^3$ (22) will become

$$
\begin{aligned}
& y_{n+1} - y_n = a_1 h + a_2(2x_n h + h^2) + a_3(3x_n^2 h + 3x_n h^2 + h^3) \\
& + a_4 e^{\lambda x_n}(e^{\lambda h} - 1) + b((\cos x_n \cosh - \sin x_n \sinh) - \cos x_n)
\end{aligned}
\qquad \textbf{(23)}
$$

If we now substitute for various values of a_1, a_2, a_3, a_4 and b in Equation (23), we have the following set of our new numerical scheme.

$$
\begin{aligned}
y_{n+1} - y_n = & \left\{
\begin{array}{l}
f_n - \left[\begin{array}{l} f_n^1 - \left(f_n^2 - (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda} + x_n) + f_n^4\right)x_n - \\ (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda^2} + \tfrac{x_n}{\lambda} + \tfrac{x_n^2}{2}) - f_n^4 ATAN(x_n) \end{array}\right] x_n - \\
\dfrac{1}{2}\left(f_n^2 - (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda} + x_n) + f_n^4\right)x_n^2 - \\
(f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda^3} + \tfrac{x_n}{\lambda^2} + \tfrac{x_n^2}{2\lambda} + \tfrac{x_n^3}{3}) - f_n^4
\end{array}
\right\} h + \\
& \frac{1}{2}\left[\begin{array}{l} f_n^1 - \left(f_n^2 - (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda} + x_n) + f_n^4\right)x_n - \\ (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda^2} + \tfrac{x_n}{\lambda} + \tfrac{x_n^2}{2}) - f_n^4 ATAN(x_n) \end{array}\right](2x_n h + h^2) + \\
& \frac{1}{6}\left(f_n^2 - (f_n^3 + f_n^4 ATAN(x_n))(\tfrac{1}{\lambda} + x_n) + f_n^4\right)\left(3x_n^2 h + 3x_n h^2 + h^3\right) + \\
& \frac{1}{\lambda^4}\left(f_n^3 + f_n^4 ATAN(x_n)\right)e^{\lambda x_n}\left(e^{\lambda h} - 1\right) - \\
& f_n^4\left(\left(ATAN(x_n)\cos(h) - \sin(h)\right) - ATAN(x)\right)
\end{aligned}
\qquad \textbf{(24)}
$$

Equation (24) is the required scheme and it can solve problems of the form (1) effectively. Equation (24) is the new Standard Finite – difference Scheme and it is a one-step method.

Convergence of the New Standard Finite Difference Scheme

We want to establish that our new numerical scheme in Equation (24) can be expressed as a one-step method in the form of Equation (3) where $\phi(x_n, y_n; h)$ is the increment function. If we expand equation (24) and simplify further we have the following:

$$
y_{n+1} = y_n + h\left\{f_n + Bf_n^1 + C_n^2 + Df_n^3 + Ef_n^4 + Ff_n^4 - Gf_n^4\right\} \qquad \textbf{(25)}
$$

Letting B = $\frac{h}{2}$, C = $x_n^2 - x_n h + \frac{h^3}{6}$,

D = $\left(\frac{x_n^3}{6} - \frac{x_n^2 h}{4} - \frac{h^2}{\lambda^3} - \frac{h}{2\lambda^2} - \frac{x_n^2}{2\lambda} - \frac{x_n^3}{2} - \frac{x_n h}{2\lambda} - \frac{h^2}{6\lambda} - \frac{x_n h^2}{6} + \frac{1}{\lambda^3}\right)$, E = $\left(\frac{h^2}{6} - 2x_n^2 + 1\right)$

F = $a\tan(x)\left(\frac{x_n^3}{6} - \frac{x_n^2 h}{4} - \frac{h}{2\lambda^2} - \frac{h}{2} - \frac{x_n^2}{2\lambda} - \frac{x_n^3}{2} - \frac{x_n h}{2\lambda} - \frac{h^2}{2\lambda} - \frac{x_n h^2}{4}\right)$ and

G = $\frac{1}{h}\left(\left(a\tan(x_n)\cos(x_n) - \sin(h)\right) - a\tan(x_n)\right)$

Therefore the new numerical scheme (25) can be written in the form:

$$y_{n+1} = y_n + h\left\{f_n + Bf_n^1 + C_n^2 + Df_n^3 + (E + F - G)\,f_n^4\right\} \quad \textbf{(26)}$$

therefore we can write our new numerical scheme as a one-step method of this form:

$$y_{n+1} = y_n + h\phi(x_n, y_n; h), \quad \textbf{(27)}$$

where $\phi(x, y; h)$ is the increment function, as stated before in Henrici (1962) defined any algorithm for solving a differential equation of the form above as a ONE-STEP METHOD. From Equations (26) and (27) we see that:

$$\phi(x, y; h) = h\left\{f_n + Bf_n^1 + C_n^2 + Df_n^3 + (E + F - G)\,f_n^4\right\}. \quad \textbf{(28)}$$

Theorem 3

Let the increment function of the scheme (24) be defined by (27) be continuously jointly as a function of its arguments in the region defined by $x \in [a,b]$, $y \in (-\infty, \infty)$, $0 \le h \le h_0$, where $h_0 > 0$, and let there exist a constant L such that:

$$\left|\phi\left(x_n, y_n^*; h\right) - \phi\left(x_n, y_n; h\right)\right| \le L\left|y_n^* - y_n\right| \quad \textbf{(29)}$$

for all $(x_n, y_n; h)$ and $(x_n, y_n^*; h)$ in the region just defined. Then the relation $\phi(x_n, y_n, 0) = f(x_n, y_n)$ is a necessary and sufficient condition for the convergence of our method (24). Considering the above theorem we can write Equation (28) as:

$$\phi(x_n, y_n; h) = \begin{Bmatrix} f(x_n, y_n) + Bf^1(x_n, y_n) + Cf^2(x_n, y_n) + \\ Df^3(x_n, y_n) + (E + F - G)\,f^4(x_n, y_n) \end{Bmatrix} \quad \textbf{(29)}$$

$$\phi(x_n, y_n^*; h) = \begin{Bmatrix} f(x_n, y_n^*) + Bf^1(x_n, y_n^*) + Cf^2(x_n, y_n^*) + \\ Df^3(x_n, y_n^*) + (E + F - G)\,f^4(x_n, y_n^*) \end{Bmatrix} \quad \textbf{(30)}$$

Thus we have:

$$\phi(x_n, y_n^*; h) - \phi(x_n, y_n; h) = \begin{Bmatrix} f(x_n, y_n^*) - f(x_n, y_n) + Bf^1(x_n, y_n^*) - f^1(x_n, y_n) + \\ Cf^2(x_n, y_n^*) f^2(x_n, y_n) + Df^3(x_n, y_n^*) - f^3(x_n, y_n) + \\ (E + F - G) f^4(x_n, y_n^*) f^4(x_n, y_n) \end{Bmatrix} \quad \textbf{(31)}$$

Let E+F-G = K, then we have:

$$\phi(x_n, y_n^*; h) - \phi(x_n, y_n; h) = \begin{Bmatrix} f(x_n, y_n^*) - f(x_n, y_n) + \\ Bf^1(x_n, y_n^*) - f^1(x_n, y_n) + \\ Cf^2(x_n, y_n^*) f^2(x_n, y_n) + \\ Df^3(x_n, y_n^*) - f^3(x_n, y_n) + \\ Kf^4(x_n, y_n^*) f^4(x_n, y_n) \end{Bmatrix} \quad \textbf{(32)}$$

Let $\bar{y}_n$ be defined as a point in the interior of the interval whose ends are y and y^*, if we apply the mean value theorem, we have:

$$\begin{aligned} f(x_n, y_n^*) - f(x_n, y_n) &= \frac{\partial f(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) \\ f^{(1)}(x_n, y_n^*) - f^{(1)}(x_n, y_n) &= \frac{\partial f^{(1)}(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) \\ f^{(2)}(x_n, y_n^*) - f^{(2)}(x_n, y_n) &= \frac{\partial f^{(2)}(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) \\ f^{(3)}(x_n, y_n^*) - f^{(3)}(x_n, y_n) &= \frac{\partial f^{(3)}(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) \\ f^{(4)}(x_n, y_n^*) - f^{(4)}(x_n, y_n) &= \frac{\partial f^{(4)}(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) \end{aligned} \quad \textbf{(33)}$$

If we define,

$$\begin{aligned} L &= \underset{(x_n, y_n) \in D}{Sup} \frac{\partial f(x_n, y_n)}{\partial y_n} \\ L_1 &= \underset{(x_n, y_n) \in D}{Sup} \frac{\partial f^{(1)}(x_n, y_n)}{\partial y_n} \\ L_2 &= \underset{(x_n, y_n) \in D}{Sup} \frac{\partial f^{(2)}(x_n, y_n)}{\partial y_n} \end{aligned} \quad \textbf{(34)}$$

$$L_3 = \underset{(x_n, y_n)\in D}{Sup} \frac{\partial f^{(3)}(x_n, y_n)}{\partial y_n}$$

$$L_4 = \underset{(x_n, y_n)\in D}{Sup} \frac{\partial f^{(4)}(x_n, y_n)}{\partial y_n}$$

If we put the relations above in equation (32) we have:

$$\phi(x_n, y_n^*; h) - \phi(x_n, y_n; h) = \begin{Bmatrix} \frac{\partial f(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) + \\ B\frac{\partial f^1(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) + \\ C\frac{\partial f^2(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) + \\ D\frac{\partial f^3(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) + \\ K\frac{\partial f^4(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) \end{Bmatrix} \quad \textbf{(35)}$$

$$= \underset{(x_n, y_n)\in D}{Sup} \frac{\partial f(x_n, y_n)}{\partial y_n}\left(y_n^* - y_n\right) + B \underset{(x_n, y_n)\in D}{Sup} \frac{\partial f^1(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) +$$
$$C \underset{(x_n, y_n)\in D}{Sup} \frac{\partial f^2(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) + D \underset{(x_n, y_n)\in D}{Sup} \frac{\partial f^3(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right) + \quad \textbf{(36)}$$
$$K \underset{(x_n, y_n)\in D}{Sup} \frac{\partial f^4(x_n, \bar{y})}{\partial y_n}\left(y_n^* - y_n\right)$$

Therefore we have the following:

$$\phi(x_n, y_n^*; h) - \phi(x_n, y_n; h) = \begin{Bmatrix} L\left(y_n^* - y_n\right) + BL_1\left(y_n^* - y_n\right) + \\ CL_2\left(y_n^* - y_n\right) + DL_3\left(y_n^* - y_n\right) + \\ KL_4\left(y_n^* - y_n\right) \end{Bmatrix} \quad \textbf{(37)}$$

$$\phi(x_n, y_n^*; h) - \phi(x_n, y_n; h) = \left(L + BL_1 + CL_2 + DL_3 + KL_4\right)\left(y_n^* - y_n\right) \quad \textbf{(38)}$$

If take the absolute value of both sides of (38) we have:

$$\left|\phi(x_n, y_n^*; h) - \phi(x_n, y_n; h)\right| \le \left|L + BL_1 + CL_2 + DL_3 + KL_4\right|\left|y_n^* - y_n\right| \quad \textbf{(39)}$$

Let M = $\left|L + BL_1 + CL_2 + DL_3 + KL_4\right|$ **(40)**

then we have $\left|\phi(x_n, y_n^*; h) - \phi(x_n, y_n; h)\right| \le M\left|y_n^* - y_n\right|$ **(41)**

Therefore, we can say that our new scheme also converge.

Consistency of the New Standard-Finite Difference Scheme

Definition (Fatunla, 1988): A numerical scheme with an increment function of $\phi(x_n, y_n; h)$ is said to be consistent with the initial value problem (1) if:

$$\phi(x_n, y_n; 0) = f(x, y) \tag{42}$$

A consistent method has order of at least one. We say our new numerical scheme is consistent since equation (24) reduces to (42) when h = 0, from equation (26). We therefore say that the method is consistent.

Stability Analysis of the New Scheme: Recall in from Theorem 2, Let $y_n = y(x_n)$ and $l_n = l(x_n)$ denote two different numerical solutions of differential equations (1) with the initial conditions specified as $y(x_0) = \eta$ and $l(x_0) = \eta^*$ respectively, such that $\left|\eta - \eta^*\right| < \varepsilon, \varepsilon > 0$. If the two numerical estimates are generated by the integration scheme (24), we have:

$$y_{n+1} = y_n + h\phi(x_n, y_n; h) \tag{43}$$

$$l_{n+1} = l_n + h\phi(x_n, l_n; h) \tag{44}$$

The condition that:

$$\left|y_{n+1} - l_{n+1}\right| \le K\left|\eta - \eta^*\right| \tag{45}$$

is the necessary and sufficient that our new method (24) be stable and convergent.

Proof: Let,

$$y_{n+1} = y_n + h\begin{Bmatrix} f(x_n, y_n) + Bf^1(x_n, y_n) + Cf^2(x_n, y_n) + \\ Df^3(x_n, y_n) + Kf^4(x_n, y_n) \end{Bmatrix} \tag{46}$$

$$l_{n+1} = l_n + h\begin{Bmatrix} f(x_n, l_n) + Bf^1(x_n, l_n) + Cf^2(x_n, l_n) + \\ Df^3(x_n, l_n) + Kf^4(x_n, l_n) \end{Bmatrix} \tag{47}$$

then,

$$y_{n+1} - l_{n+1} = y_n - l_n + h\begin{Bmatrix} f(x_n, y_n) - f(x_n, l_n) + B\left(f^1(x_n, y_n) - f^1(x_n, l_n)\right) + \\ C\left(f^2(x_n, y_n) - f^2(x_n, l_n)\right) + \\ D\left(f^3(x_n, y_n) - f^3(x_n, l_n)\right) + K\left(f^4(x_n, y_n) - f^4(x_n, l_n)\right) \end{Bmatrix} \tag{48}$$

Applying mean value theorem as before, we have,

$$y_{n+1} - l_{n+1} = y_n - l_n + h\left\{\begin{array}{l} \frac{\delta f(x_n, l_n)}{\delta l_n}(x_n - l_n) + B\frac{\delta f^1(x_n, l_n)}{\delta l_n}(x_n - l_n) + \\ C\frac{\delta f^2(x_n, l_n)}{\delta l_n}(x_n - l_n) + \\ D\frac{\delta f^3(x_n, l_n)}{\delta l_n}(x_n - l_n) + K\frac{\delta f^4(x_n, l_n)}{\delta l_n}(x_n - l_n) \end{array}\right\} \quad \textbf{(49)}$$

$$y_{n+1} - l_{n+1} = y_n - l_n + h\left\{\begin{array}{l} L(x_n, l_n) + BL_1(x_n, l_n) + \\ CL_2(x_n, l_n) + L_3(x_n, l_n) + KL_4(x_n, l_n) \end{array}\right\}$$

$$y_{n+1} - l_{n+1} = y_n - l_n + h\{L + BL_1 + CL_2 + L_3 + KL_4\}(x_n - l_n) \quad \textbf{(51)}$$

Taking absolute value of both sides of (51) gives:

$$|y_{n+1} - l_{n+1}| \le |y_n - l_n| + h|L + BL_1 + CL_2 + L_3 + KL_4||x_n - l_n|$$

Let H = $h|L + BL_1 + CL_2 + L_3 + KL_4|$ and $y(x_0) = \eta$, $l(x_0) = \eta*$, given $\varepsilon > 0$

Then $|y_{n+1} - l_{n+1}| \le H|x_n - l_n|$ **(52)**

$|y_{n+1} - l_{n+1}| \le H|\eta - \eta^*|$ for every $\varepsilon > 0$.

Then we conclude that our method (24) is stable and hence convergent.

CONCLUSION

We have established that the scheme is convergent, consistent and stable; it shows that the numerical solution of the scheme will converge competitively with the exact (theoretical) solution especially as the step-length tends to zero.

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