

Derivation of Expressions for Apparent Conductivity and Wave Impedance in an Inhomogenous Earth Medium.

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ABSTRACT

Electromagnetic waves are produced in the earth medium from an induced magnetotelluric current due to the Earth's magnetic field. The direction of the induced magnetotelluric current is the same as the plane propagating electromagnetic waves which is perpendicular to the electric and magnetic fields.

Delineating extensively the relevant fundamental or Maxwell's equations and the equations of plane propagating electromagnetic waves and assuming an isotropic homogenous earth medium, expressions have been derived for the impedance and apparent resistivity with apparent conductivity in a homogenous earth medium.

The most important parameter in studying the electrical flow of current and other practical with paramount endogenous earth behavior is the apparent conductivity or apparent resistivity which has been elaborated.

(Keywords: apparent conductivity, impedance, homogenous isotropic earth medium, electromagnetic waves, magneto telluric current)

INTRODUCTION

A rapidly changing electric field produces a magnetic field. The changing electric field and magnetic fields then interact with earth other to produce an electromagnetic field and the two fields are interwoven. A magnetic field is created when an electric field is passed through a coil, magnetic field between the poles of a permanent magnetic.

Electric and magnetic fields that do not vary with time and independently existing on the atomic scale are considered as electrostatic and magnetostatic fields. Conduction in rocks takes place through minerals contained in pores of rocks near the earth's surface. The conducting minerals must be high enough in concentrations, to change appreciably the electrical properties of rocks in which they occur. These conducting minerals comprise magnetite, hematite, carbon, graphite, pyrite, and pyrohotite which occur occasionally in sufficient quantity to foster conduction. Thus, most rocks are electrolytic conductors and propagation of current is by ionic conduction and conductivity of the rocks which vary with the mobility concentration and degree of dissociation of the ions of the mineral solutions. The conductivity of a porous rock varies with the volume and arrangements of the pores and even the amount of conducting water contained by Archie's law:

$$\rho = a\phi^{-m} s^{-n} \rho_{\sigma}$$

where ρ = bulk resistivity of the rock

ϕ = fractional pore volume (porosity)

S = fractional of the pore containing water

ρ_{σ} = water resistivity

$n = 2$

$a, m = \text{constants with } 0.5 < a < 2.5; 1.3 < m < 2.5$

An electric field is obviously produced along the surface of the Earth by a result of electrical current flow. The natural electrical potential developed in the earth as a result of current flow due to electrochemical action between minerals and the solutions with which they are contact is called the Self Potential.

Natural electric potential is also due to magnetic – telluric current which is the current induced in the

Earth by the time varying portion of the Earth's magnetic field, and Audio Frequency Magnetic Field (AFMAG).

The relative capabilities of materials to conduct electricity when a voltage is applied are expressed as conductivities and the resistance a material offers to the flow of current is expressed as resistivity defined by the expression:

$$\rho = \vec{E} \cdot \vec{j}$$

and conversely

$$\vec{j} = \sigma \vec{E} \text{ for conductivity,}$$

\vec{E} is the electric field intensity in volts per metre (V/m);

\vec{j} is the current density in a Ampere per square metre (A/m^2).

The unit of resistivity ρ is ohm – metre (Ωm).

The magneto-telluric field component is measured with a pair of electrodes with spacing of the order of a kilometer, hence the unit is milli volt per kilometer.

The study of electrical resistivity from vertically inhomogeneous earth wave conductivity changes continually with depth (Abramovici et al, 1978) has been extensively investigated. The study of the DC resistivity soundings on a model Earth with transition layer was made by Mallick et al. (1968), Mallick (1970), Patellic (1971), Koefold (1979), and Benerjere et al. (1980).

Models of conductivity and resistivity varying linearly for magnetotelluric (MT) Soundings was also investigated by Mallick (1970) Abramovic (1974), Rankin et al. (1975) and Kao (1982).

The computation is of magnetotelluric impedance from surface measurement of the magnetotelluric fields exhibit anisotropy, which may be due to vertically inhomogeneous structures with different electrical properties. It is assumed that the primary, natural electromagnetic fields are plane waves and the distance. Between the measures electrodes is small relative to the dimensions of the structures.

METHODOLOGY

Expressions for the impedance, apparent resistivity, and apparent conductivity are determined analytically with application of basic fundamental principles of electromagnetic wave propagation in a magneto telluric homogenous earth medium and relevant equations.

It is obvious that the apparent conductivity in the earth media can be evaluated from relevant physical parameters carefully selected as the most important electrical parameter.

DISCUSSION

The behavior of electric and magnetic field variations generated by the magnetotelluric field would be explained with Maxwell's equations which take into consideration the existence of induction currents since the earth's magnetic field is time varying and there could be induction currents.

The Maxwell's relations are;

$$\nabla \wedge \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (1)$$

$$\nabla \wedge \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t} \quad (2)$$

$$\nabla \cdot \vec{B} = 0 \quad (3)$$

Where \vec{E} is the electric field vector,
 \vec{H} is the induced magnetic fields vector,
 \vec{B} is the magnetic field vector
 \vec{D} is the displacement vector.

The electrical parameters of the medium are; μ , ϵ and ρ , representing magnetic permeability dielectric constant and electrical resistivity.

$$\vec{D} = \epsilon \vec{E} \quad (4)$$

$$\vec{j} = \frac{\vec{E}}{\rho} \quad (5)$$

$$\vec{B} = \mu \vec{H} \quad (6)$$

Substitution of the above expressions in the Maxwell equations, it can be deduced that:

$$\nabla \times \nabla \times \vec{E} = -\mu \nabla \times \frac{\partial \vec{H}}{\partial t} \quad (7)$$

$$\frac{\mu}{\rho} \frac{\partial \vec{E}}{\partial t} - \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} = -\mu \nabla \times \frac{\partial \vec{H}}{\partial t} \quad (8)$$

$$\nabla \times \nabla \times \vec{E} = \frac{\mu}{\rho} \frac{\partial \vec{E}}{\partial t} + \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} \quad (9)$$

which yields,

$$\nabla^2 \vec{E} = \frac{\mu}{\rho} \frac{\partial \vec{E}}{\partial t} + \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} \quad (10)$$

unless there are current sources or changes in the electrical properties of the medium;

$$\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E} + \nabla \cdot \nabla \vec{E} = -\nabla^2 \vec{E}$$

as

$$\nabla(\nabla \cdot \vec{E}) = 0$$

Considering a one dimensional case earth model, the electric field is along the - axis of the direction of wave propagation is along the X-axis. The Z-axis will be normal to the interface with the magnetic field.

The propagation of electromagnetic wave in a medium consisting of indirectly alternatively layers could also be considered, where we chose an arbitrary part of neighboring coordinates on their interfaces, layer 1 extends from Z = 0 to Z = h₁ and layer 2 from Z = -h₂ to Z = ∞; the equation for the field are determined with the Maxwell's equations.

The differential equation for the electric field from Equation 10 above is given below as;

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \frac{\mu}{\rho} \frac{\partial E_x}{\partial t} + \mu \epsilon \frac{\partial E_x}{\partial t} \quad (11)$$

along the x-axis

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \quad (12)$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \frac{\mu}{\rho} \frac{\partial E_z}{\partial t} + \mu \epsilon \frac{\partial^2 E_z}{\partial t^2} \quad (13)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \left(\frac{1}{\rho} + i\omega \epsilon \right) E \quad (14)$$

and

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega \mu H_z \quad (15)$$

and similarly \vec{H} could replace \vec{E} to get its three components.

The H and E polarized waves along the y – direction, respectively are as follows below.

Derivation of Expressions for Impedance and Apparent Conductivity

The current sheets induced in the earth by a varying magnetic field must be parallel to the earth surface, since no appreciable amount of current unless there are regional changes in resistivity. It is possible to select a certain coordinate system such that only one component of the electric field is zero in our particular case. With the choice of coordinate and no current flow or electric field in the y or z directions:

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\mu}{\rho} \frac{\partial E_x}{\partial t} + \mu \epsilon \frac{\partial^2 E_x}{\partial t^2} \quad (15)$$

i.e., using Equation 10:

Assuming a periodic solution;

$$E_x \geq Ae^{i\omega t z} \quad (16)$$

Hence; substituting into the above Expression 15 gives:

$$\gamma = \pm \left(\frac{i\omega\mu}{\rho} - \epsilon\mu\omega \right)^{\frac{1}{2}} \quad (17)$$

Where γ is the wave number which is the reciprocal of a radian wavelength. Resistivity is extremely important at very low frequencies considered in magnetotelluric method more than the dielectric constant or the magnetic permeability in determining variations in the wave number.

$$\begin{aligned} \mu &= 12.56 \times 10^{-7} \text{ Hm}^{-1} \text{ and } \epsilon = 10^{-6} \text{ Fm}^{-1} \\ \gamma^2 &= (i\omega \times 12.56 \times 10^{-8} - \omega^2 \times 12.56 \times 10^{-3}) \end{aligned} \quad (18)$$

As an example for a highly conductive sedimentary rock for which resistivity is 100mm-ohm.

$$\begin{aligned} \gamma^2 &= |\gamma^2| e^{i\omega} \text{ is a complex quantity.} \\ \gamma &= \gamma e^{i\omega/2} \end{aligned} \quad (19)$$

The magnitude of the wave number is:

$$|\gamma^2| = (12.56 \times 10^{-8})^2 + (12.56 \times 10^{-13})^2 \quad (20)$$

and the phase is:

$$\Psi/2 = \arctan \frac{12.56 \times 10^{-13} \omega^2}{12.56 \times 10^{-8} \omega} \quad (21)$$

With the very low frequency considered in magnetotelluric field, the first term in the above equation exceeds the second order of 10^{-4} , hence;

$$\gamma = (i\omega\mu / \rho)^{\frac{1}{2}} \quad (22)$$

$$\gamma = 1 + i/\sqrt{2} (\omega\mu / \rho)^{\frac{1}{2}} \quad (23)$$

As a linear combination of solutions to the differential equation in E_x ;

$$E_x = Ae^{i\omega t + \gamma z} + Be^{i\omega t - \gamma z} \quad (24)$$

and the arbitrary constants A and B must be evaluated.

The current generates a time – varying magnetic field;

$$H_y = H_o e^{i\omega t} \quad (25)$$

$$\frac{\partial H_y}{\partial t} = i\omega H_o e^{i\omega t} \quad (26)$$

$$= i\omega H_y \quad (27)$$

From the Maxwell relation;

$$\text{Curl} E = -i\omega\mu H_y \quad (28)$$

$$\frac{\partial E_x}{\partial z} = -i\omega\mu H_y \quad (29)$$

$$\frac{\partial E_x}{\partial z} = \gamma (Ae^{i\omega t \gamma z} + Be^{i\omega t - \gamma z}) \quad (30)$$

$$= i\omega\mu H_y \text{ via differentiation.} \quad (31)$$

Hence;

$$H_y = \gamma / i\omega\mu (Ae^{i\omega t \gamma z} + Be^{i\omega t - \gamma z}) \quad (32)$$

2 The wave impedance Z is thus;

$$Z = \frac{E_x}{H_y} = -i\omega\mu / Ae^{\gamma z} + Be^{-\gamma z} / Ae^{\gamma z} - Be^{-\gamma z} \quad (33)$$

With the identity $\sqrt{A/B} = e \ln \sqrt{A/B}$ we have;

$$Z = i\omega\mu / \gamma \exp(\ln \sqrt{A/B} + \gamma z) + \exp(\ln \sqrt{A/B} - \gamma z). \quad (34)$$

$$= i\omega\mu / \gamma \coth(\gamma z + \ln \sqrt{A/B}). \quad (35)$$

The constants A and B can be eliminated by considering initial or boundary conditions or also by considering ratios of wave impedance at two different positions.

With the second approach;

$$\ln \sqrt{A/B} = \coth^{-1}(\gamma z / i\omega\mu) - \gamma z \quad (36)$$

and considering another position

$$Z_2 = i\omega\mu / \gamma \coth(\gamma z_2 + \ln \sqrt{A/B}) \quad (37)$$

$$= i\omega\mu / \gamma \coth(\gamma(z_2 - z_1) + \coth^{-1}(\gamma z_1) / i\omega\mu) \quad (38)$$

Which is valid only if the two positions z_1 and z_2 are in the same medium. In a layered medium, they have to be in the same direction.

The value of the wave impedance at the surface related to the wave impedance at the bottom of the first layer is as follows;

$$Z_2(z=0) = i\omega\mu / \gamma \coth(\gamma z_1 - \coth^{-1}(\gamma(z=h) / i\omega\mu)) \quad (39)$$

The simplest case is a completely homogenous earth. In this case, the thickness of the first layer becomes infinite; as this happens, the hyperbolic cotangent approaches unity,

$$Z_o = i\omega\mu = (i\mu\rho)^{\frac{1}{2}} \quad (40)$$

Solving this equation;

$$\rho_a = -iz^2 / \omega\mu = i / \omega\mu (E)^2 / (H)^2 \quad (41)$$

Thus, the apparent conductivity,

$$\sigma_a = \frac{i\mu\omega}{z^2} \quad (42)$$

The imaginary forms ρ_a and σ_a indicate that there is a 45° phase between the oscillations in the magnetic and electric field intensities.

CONCLUSION

The most important property in studying the electrical current flow in earth medium is the apparent conductivity. An expression has been deduced for evaluating the impedance. Hence the apparent resistivity and conductivity can be deduced.

The imaginary forms in which ρ_a and σ_a appear indicate a 45° phase difference in the oscillations of the magnetic and electric field intensities.

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