Application of the "Bang Bang" Theory in Auto-Control of Drones.

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ABSTRACT

The "Bang-Bang" theory is discussed. Its application in auto-control of drones is highlighted. A generic graphical solution of a minimum-time control problem, which results in a "Bang-Bang" theory, is presented. A similar minimum-time mathematical control problem, modeled after a "source" and "target," is posed and solved. The solution shows the path that could be taken to ensure minimum time to accomplish a set objective.

(Keywords: Bang Bang theory, minimum-time, trajectory, auto-control, drone, target)

INTRODUCTION

Whenever there is an interest to hit a target, either locally or remotely, it has to be done in the fastest possible time before the target changes its position. This kind of situation occurs very often in military activities. The focus of this paper will be on remote warfare, such as in the use of drones (controlled quite some distance away), employed to hit targets of interest remotely.

By observation, it will be immediately noticed that this is a minimum-time optimal control problem. The "Bang-Bang" theory in optimal control was developed based on minimum time (Craggs 2008). The possibility of using that theory in the auto-control of drones, to hit targets precisely, will now be explored.

THE "BANG-BANG" THEORY

In the "Bang-Bang" theory, the control force u(t) is permitted to change from +1 to –1 or from –1 to +1, at most once. Mathematically, that means:

$$u(t) = +1 \quad \textbf{(1)}$$

The trajectory of u(t) will hence be defined within those boundaries. One is then free to start the optimum trajectory with u(t) = +1 or –1. The dynamic states of the object that moves along the trajectory is defined by:

$$\dot{\chi}_2(t) = u(t) \quad \textbf{(2)}$$

and

$$\dot{\chi}_1(t) = \chi_2(t) \quad \textbf{(3)}$$

Putting Equations (2) and (3) in a state-matrix form, we have:

$$\begin{pmatrix} \dot{\chi}_1(t) \\ \dot{\chi}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \chi_1(t) \\ \chi_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \quad \textbf{(4)}$$

that is, χ(t) = Ax (t) + Bu(t) **(5a)**

$$\text{where, } \dot{\chi}(t) = \begin{cases} \dot{\chi}_1(t) \\ \dot{\chi}_2(t) \end{cases}$$

$$\chi(t) = \begin{cases} \chi_1(t) \\ \chi_2(t) \end{cases}, \text{ a 2 x 1 state vector}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ a 2 x 2 matrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ a 2 x 1 matrix}$$

Now, let the time that the object to be transported being at its initial position, be to (initial time, in seconds), and the time it is to arrive at its final destination be tf (final time, in seconds).

With those definitions, solutions will now be found to the state equations, eqns. (2) and (3).

Since u(t) = + 1 (recall Equation (1)), a value of +1 will be assigned to u(t). Hence, let u(t) = 1.

$$\therefore \quad \text{From Eqn. (2), } \dot{\chi}_2(t) = 1, \qquad \textbf{(5b)}$$

$$\text{and} \int_{x_2(t_o)}^{\chi_2(t_f)} \dot{\chi}_2(t) = \int_{x_2(t_o)}^{\chi_2(t_f)} d\chi_2(t) = \int_{t_o}^{t_f} dt_f \qquad \textbf{(6)}$$

$$\text{giving } 0 - \chi_2(t) = t_f - t_o$$

$$\chi_2(+) = (t_o - t_f) < 0 \qquad \textbf{(7)}$$

$$\text{But } \dot{\chi}_1(t) = \chi_2(+), \text{ (recall Eqn. (3))}$$

Hence from Eqn. (7),

$$\dot{\chi}_1(t) = \chi_2(t) = (t_o - t_f) < 0 \qquad \textbf{(8)}$$

On integration, Eqn. (8) becomes:

$$\int_{\chi_1(t)}^{0} d\chi_1(t) = \int_{t_o}^{t_f} (t_o - t_f)\, dt \qquad \textbf{(9)}$$

$$\therefore \quad -\chi_1(t) = -\tfrac{1}{2}(t_o - t_f)^2 \qquad \textbf{(10)}$$

Substituting Eqn. (8) into (10), we have:

$$\chi_1(t) = \tfrac{1}{2}(x_2(t))^2 > 0 \qquad \textbf{(11a)}$$

Similarly, by setting $u(+) = -1$,

$$\dot{\chi}_2(+) = u(+) = -1, \qquad \textbf{(11b)}$$

$$\text{and} \int_{x_2(t_o)}^{\chi_2(t_f)} \dot{\chi}_2(t) = \int_{x_2(t_o)}^{\chi_2(t_f)} d\chi_2(t) = \int_{t_f}^{t_o} dt_f \qquad \textbf{(12)}$$

$$\text{giving: } 0 - \chi_2(t) = t_o - t_f$$

$$\chi_2(t) = (t_f - t_o) > 0 \qquad \textbf{(13)}$$

$$\text{But } \dot{\chi}_1(t) = \chi_2(t), \text{ (recall Eqn. (3))}$$

Hence from eqn. (13),

$$\dot{\chi}_1(t) = \chi_2(t) = (t_f - t_o) > 0 \qquad \textbf{(14)}$$

On integration, equation (14) becomes

$$\int_{\chi_1(t)}^{0} d\chi_1(t) = \int_{t_o}^{t_f} (t_f - t_o)\, df \qquad \textbf{(15)}$$

$$-\chi_1(t) = \tfrac{1}{2}(t_o - t_f)^2 \qquad \textbf{(16)}$$

$$\chi_1(t) = -\tfrac{1}{2}(\chi_2(t))^2 < 0 \qquad \textbf{(17)}$$

It will be noticed that Eqns. (13) and (17) are parabolic (Smyrl 2009). A plot of the two Equations ((13) and (17)) is shown in Figure 1. The curve AOB is known as the switching curve.

Figure 1 illustrates a plot of two trajectories χ_1 (t) and χ_2 (t) as given by Eqns. (13) and (17); defining the possible paths of an object in motion.

Interpretation of the Trajectories in Figure 1

The ultimate aim of the object in motion, characterized by Equations (13) and (17), is to arrive at the point defined in the two axes by (0, 0), that is the origin in the shortest possible time. In order to meet this objective, the object must move along the parabola characterized by u (t) = +1 or u (t) = -1, which passes through the zero point (0, 0). However, since only one switch-over is possible, the field of optimal trajectories must follow the pattern shown in Figure 1.

By any movement characterized by u (t) = +1 or u (t) = -1, the object in motion follows either of the parabolic directions passing through zero until it reaches the parabola with u (t) = -1 or u (t) = +1, and ultimately arrives at the zero point (0, 0).

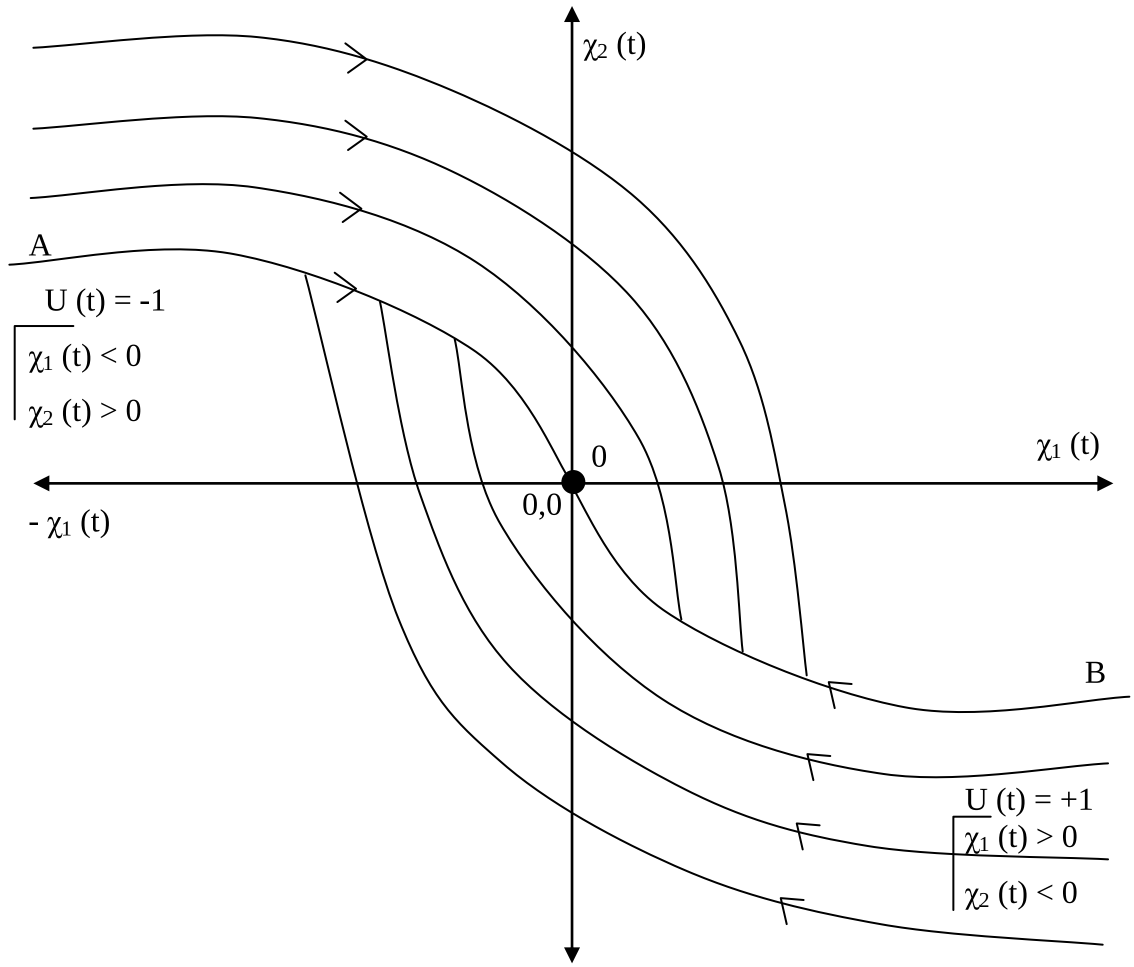

Figure 1: A Plot of the Two Trajectories, χ1 (t) and χ2 (t) as Obtained from Eqns. (13) and (17).

This must be accomplished in the shortest possible time.

Auto-Control and Activation of the Drone

The drone is an aircraft (mostly used by the military) that is remotely and automatically controlled (Wikipedia). The auto-pilot sits at the ground station, distance away from the drone which flies at pre-determined altitude, controlling the flight. The geographical co-ordinates of the target of interest are remotely fed to the drone, and activated to release its weapon; to hit a target. Its weapons sometimes do miss the intended target widely or narrowly (Washington Post).

One of the remedies that could eliminate this wide and narrow miss could be to model the direction of movement of the drone's weapon after the trajectories defined by Equations (13) and (17). The ultimate destination of the weapon would be the defined target, characterized by the point (0, 0) in Figure 1.

An Illustration of the "Bang-Bang" Theory with a Mathematical Problem

A dynamical system is subject to the equations χ_1 (t) = χ_2 (t), χ_2 (t) = u (t), where |u (t)| ≤ 1. The objective is to determine the minimum time required to move from conditions:

$$(\chi_1 (t), \chi_2 (t)) = (a, b) \quad \textbf{(18)}$$

to $(\chi_1(t), \chi_2(t)) = (0, 0)$ **(19)**

Here, a, b are equivalent to deviations from the set target of 0, 0 in the drone target objective.

Solution: The solution corresponding to u (t) = +1 is given by:

$$\chi_2(t) = t + \alpha \quad \textbf{(20)}$$

$$\chi_1(t) = \beta + ½(t + \alpha)^2 = \beta + ½\chi_2^2(t) \quad \textbf{(21)}$$

In a plane with co-ordinates $(\chi_1(t), \chi_2(t))$, Equation (21) represents a family of parabolas with horizontal axis and vertex to the left. Similarly, u (t) = -1 leads to:

$$\chi_1(t) = \beta - ½\chi_2^2(t) \quad \textbf{(22)}$$

Equation (22) also leads to a family of parabolas with vertex to the right. These are both illustrated in Figure 2.

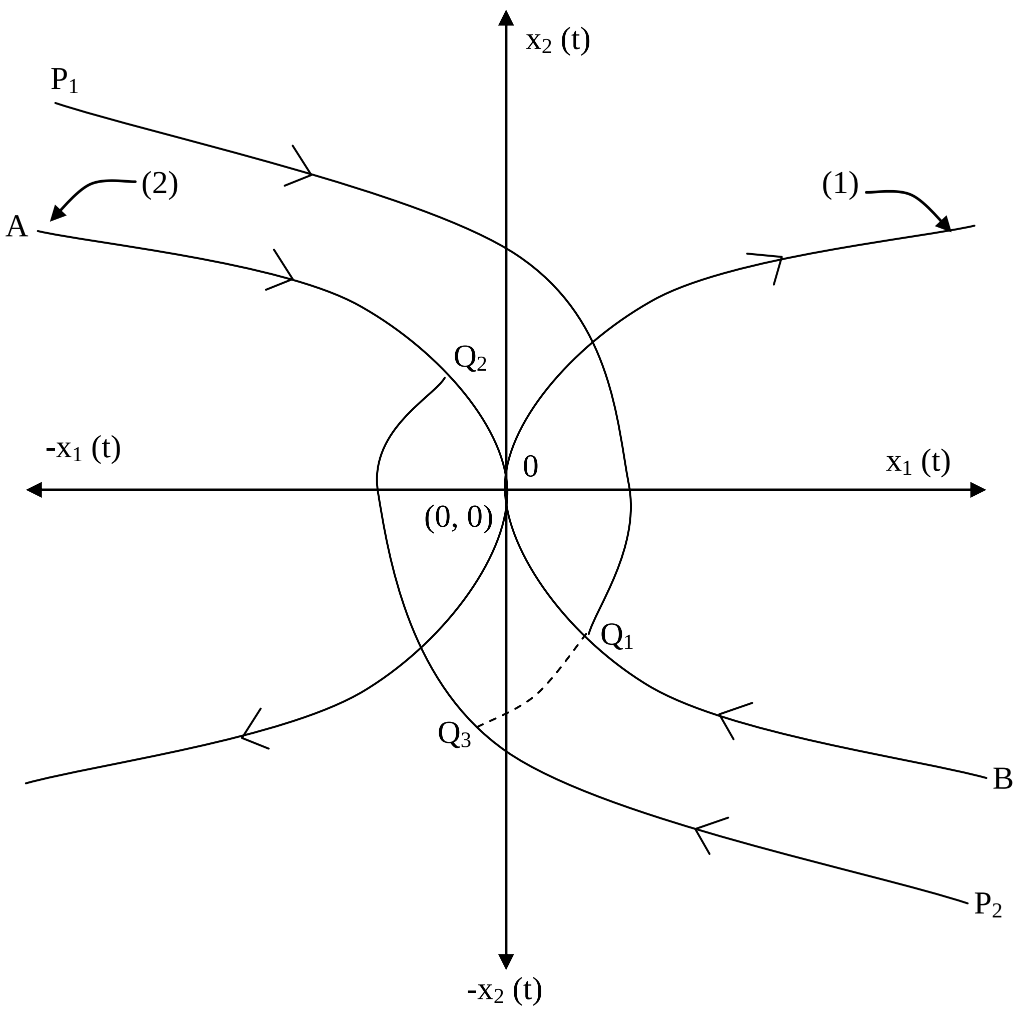

Figure 2: A Typical Sketch from Eqns. (21) and (22).

Now, as the time, t, increases, the point x_1 (t) and x_2 (t) denoted by P_1 moves in the direction of the arrow. To get to the origin (the target), that is: (x_1 (+), x_2 (t)) (0, 0), the point must approach the origin on the curve AO or OB. Similar conditions apply to the point P_2. The point (P_2) must approach the origin on the curve OB or OA.

The minimum-time solution with initial conditions represented by P_1, will be an arc P_1Q_1 followed by arc Q_1O. For a point to the left, like P_2, the solution is represented by arc P_2Q_2 and Q_2O. The curve AOB here is also known as the "switching curve."

DISCUSSION

The minimum-time problem occurs in very many areas in life. Objectives like steering a ship in the correct direction to its anchor point against surging waves, as fast as possible to avoid driftage by the waves, is a minimum-time control problem. Auto altitude control of an aircraft (to avoid loss of altitude) is also a minimum-time control problem. If the altitude is modeled as the set target, the Bang-Bang principle could stabilize an aircraft at a fixed altitude until the pilot forces it to change during landing preparations.

The "Bang-Bang" theory, as presented here, is a generic "tool" for solving most minimum-time control problems. As long as the initial states of a system, its intended path and final co-ordinates are clearly defined (mathematically), then the theory can be applied successfully.

CONCLUSION

The objective of the paper was to demonstrate how the "Bang-Bang" theory could be applied in optimal control in auto-control of drones; to enable them hit targets precisely. This, by itself, is a minimum-time control problem. Figure 1 illustrates graphically how such an objective could be met in minimum-time. As long as the objectives, the intended trajectories and the co-ordinates of the target are well-defined and properly coded into a computer, the set objective(s) could be achieved in minimum-time. A similar mathematical optimal control problem (and its solution) further illustrated the application of the "Bang-Bang" theory.

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SUGGESTED CITATION

Itaketo, U.T. 2011. "Application of the "Bang Bang" Theory in Auto-Control of Drones". *Pacific Journal of Science and Technology*. 12(1):25-29.

Pacific Journal of Science and Technology