The Vibration of a Self-Gravitating Sphere.

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ABSTRACT

The free vibration of a self-gravitating sphere consisting of an incompressible homogeneous fluid is investigated. The period of oscillation was theoretically found to be approximately ninety-four minutes. Observations from the earthquake in Chile of May 22, 1960 showed that the period of free oscillation was about fifty-four minutes. The discrepancy between the theoretical and observed estimates may be due to the neglect of elasticity in the theoretical formulation. Elasticity acts as an additional restoring force and should shorten the period.

(Keywords: self-gravitating sphere, incompressible homogeneous fluid, earthquake, elasticity, free oscillations, theoretical periods of oscillation)

INTRODUCTION

The Earth is considered as an elastic body of spherical shape. In this paper, we wish to examine its free elastic vibrations or free oscillations. In particular, we want to investigate the theoretical periods of oscillations. The first records which clearly showed the free oscillations of the Earth were obtained from the Great Chilean earthquake of May 22, 1960. To date it is the most powerful earthquake ever recorded with a magnitude of 9.5. It occurred in the afternoon 19:11 GMT, 14:11 local time (USGS, 1960) and its resulting tsunami affected southern Chile, Hawaii, Japan, the Philippines, eastern New Zealand, southeast Australia and the Aleutian Island in Alaska.

The epicenter was near Cariete some 900 km south of Santiago, with Temuco being the closest large city, while Valdivia was the most affected city. It caused localized tsunamis that severely battered the Chilean coast, with waves up to 25 meters. The main tsunami raced across the Pacific Ocean and devastated Hilo, Hawaii. Waves as high as 10.7 meters were recorded 10,000 km from the epicenter and as far away as Japan and the Philippines.

FORMULATION OF THE PROBLEM

Consider a self-gravitating sphere of radius a, consisting of an incompressible homogeneous fluid of constant density ρ. It has a mass $M = 4/3\ \pi\ G\ \rho\ a^3$, where G is the universal gravitational constant. Suppose its shape is changed slightly so that the outer surface, originally at r = a, is displaced to r = a + § where § << a. If the sphere is so deformed, it will undergo infinitesimal free oscillations. This paper attempts to investigate these oscillations.

ANALYSIS

The outer surface is r = a + ξ, where ξ is

$$§ = \sum_{l} \sum_{m} §_l^m \ 'Y_l^m \qquad \textbf{(1)}$$

write $\Phi = \Phi_0 + \Delta §$

where Φ_0 is the gravitational potential in the absence of deformation, that is,

$$\Phi_0 = -\frac{GM}{2a^3}(3a^2 - r^2),\ r < a$$

where $-\frac{GM}{R}\ \ r \geq a$

The perturbation Δ Φ must satisfy the relations, (Aki and Richards, 1980):

$\nabla^2 \Delta\Phi = 0$ everywhere, so **(2)**

$$\Delta \Phi = \sum_l \sum_m A_l^m r^l \, 'Y_l^m ; \quad r \le a \text{ and} \qquad \textbf{(3)}$$

$$\Delta \Phi = \sum_l \sum_m B_l^m r^{-l-1} \, 'Y_l^m ; \quad r \ge a \qquad \textbf{(4)}$$

A_l^m and B_l^m are determined from the jump conditions at:

$$r = a \quad [\, \Delta \Phi \,]+ = 0 \text{ and}$$

$$\left[\frac{\partial}{\partial r} \Delta \Phi \right]+ = 4 \pi G \rho \S$$

this gives $A_l^m = \dfrac{3\,G\,M}{(2\,l + 1)\, a^{l+2}} \S_l^m$ **(5)**

$$B_l^m = \frac{3\,G\,M}{(2\,l + 1)} a^{l-1} \S_l^m \qquad \textbf{(6)}$$

(Takeuchi and Saito, 1972).

The value of Φ on the deformed surface is to the first order in §,

$$\Phi_{surface} = - \frac{G\,M}{(a + \S)} + \Delta \Phi_{r = a}$$

$$\approx - \frac{G\,M}{a} \left[1 - \frac{\S}{a} \right] + \Delta \Phi_{r = a} \qquad \textbf{(7)}$$

or

$$\Phi_{surface} = - \frac{G\,M}{a} \left[1 - \sum_l \sum_m \frac{2(\,l - 1\,)}{2(\,l + 1\,)} \left(\frac{\S_l^m}{a} \right) \right] \, 'Y_l^m$$

(8)

where $\S = \sum_l \sum_m \S_l^m \, 'Y_l^m$ is the spherical harmonic expression of §. The fluid motion is a potential flow, that is:

$$U = \nabla \Psi$$

with $\nabla^2 \Psi = 0$ in $r = a$

where

$$\Psi = \sum_l \sum_m \left(\frac{r}{a} \right)^l \Psi_l^m (\,t\,) \, 'Y_l^m \qquad \textbf{(9)}$$

to determine the coefficients Ψ_l^m (t), the linearized kinematical boundary condition is applied on r = a, that is:

$$\frac{\partial \S}{\partial t} = \frac{\partial \Psi}{\partial r}$$

or $\dfrac{\partial \S_l^m}{\partial t} = \dfrac{l}{a} \Psi_l^m$ **(10)**

We also have Bernoulli's theorem, which since P = 0 on the free outer surface takes the form to

the first order

$$\frac{\partial \Psi}{\partial t} + \Phi_{surface} = \text{constant}$$

thus we get,

$$\frac{\partial \Psi_l^m}{\partial t} + \frac{2(\,l - 1\,)}{2\,(l + 1\,)} \left(\frac{g}{a} \right) \S_l^m = \text{constant} \qquad \textbf{(11)}$$

where $g = \dfrac{4}{3} \pi\, G\, a$

Combining (10) and (11), we get:

$$\frac{\partial^2 \Psi_l^m}{\partial^2 t} + \frac{2\,l\,(\,l - 1\,)}{2\,l + 1} \frac{g}{a} \Psi_l^m = 0 \qquad \textbf{(12)}$$

$$\frac{\partial^2 \S_l^m}{\partial^2 t} + \frac{2\,l\,(\,l - 1\,)}{2\,l + 1} \frac{g}{a} \S_l^m = 0 \qquad \textbf{(13)}$$

This is the equation of the motion of a simple harmonic oscillator with squared eigen-frequency and the angular velocity ω is given by:

$$\omega_l^2 = \frac{2\,l\,(\,l - 1\,)}{2\,l + 1} \frac{g}{a} = \frac{2\,l\,(\,l - 1\,)}{2\,l + 1} \left[\frac{4}{3} \pi\, G\, \rho \right] \qquad \textbf{(14)}$$

NUMERICAL COMPUTATION

Equation (14) represents the eigen-frequencies of a self-gravitating incompressible fluid sphere. There is no dependence of ω_l^2 on the order m because of spherical symmetry. There is thus a 2l + 1 degeneracy.

l = 0 is impossible for an incompressible fluid, since it implies a net change of volume.

l = 1 is a rigid body translation. Both l = 0 and l = 1 have zero frequency. The fundamental mode is l = 2 (denoted by ${}_0S_2$). For Earth-like values, the density is ρ = 5517 kg/m^3, the radius is a = 6371 km and the Universal Gravitational Constant is G = 6.67 x 10^{-11} N.m^2/kg^2.

Substituting these values in Equation (14), we have:

$$\omega_2^2 = \frac{2 \times 2\,(2-1)}{2 \times 2 + 1}\left[\frac{4}{3} \times 3.14 \times 5517 \times 6.67 \times 10^{-11}\right]$$

$$\omega_2 = \frac{111}{10^5}$$

but $\omega_2 = 2\pi f$ and the period $T = \frac{1}{f}$

$$\text{therefore, } T = \frac{2 \times 3{,}14 \times 10^5}{111 \times 60}\ \text{s}$$

$$\approx 94 \text{ minutes}$$

The period of the ${}_0S_2$ for the actual Earth is about 54 minutes. Elasticity which has been neglected in the theoretical formulation acts as an additional restoring force which should shorten the period.

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