Application of Homotopy Perturbation Method (HPM) for the Solution of Some Non-Linear Differential Equations.

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ABSTRACT

In this paper, the Homotopy Perturbation Method is implemented to solve non-linear differential equations. The HPM, was applied to the Van der pol equation and the Evolution equation, the equations are deformed into a simpler problem, which can then be easily solved. The results show that this method is effective, simple and the series converges with easily computable terms.

(Keywords: HPM, Van der pol, evolution equation, convergence)

INTRODUCTION

Most scientific problems in engineering are inherently nonlinear. Except for a few of them, a majority of nonlinear problems do not have analytical solutions. Therefore, these nonlinear equations should be solved using other methods such as numerical or perturbation methods (Fooladi et al., 2009). In the numerical method, stability and convergence should be considered so as to avoid divergence or inappropriate results each effective parameter should be solved iteratively Hoffman (1992). In the perturbation method, the small parameter is inserted in the equation. Thus, finding the small parameter and exerting it into the equation is one of the deficiencies of this method Nayfeh (1993).

HPM was first proposed by He (2000). The method does not depend on a small parameter in the equation. Using the homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a “small parameter". HPM was successfully applied to different branches of science and engineering Sadighi et al. (2009).

The main objective of this paper is to employ HPM for solving non-linear differential equations; Van der pol and the Evolution equations to be specific. The Van der pol equation was recently analyzed by Aiyesimi and Adeniyi (2008) using the Krylov-Bogolobov technique, series of assumptions were made in the study, HPM is employed here to avoid such assumptions. The evolution problem was also analyzed recently by Yanze and Krishnan (2006) using travelling wave solutions technique. The HPM approach is also employed in this study to give qualitative insight to the problem.

FUNDAMENTALS OF THE HOMOTOPY PERTURBATION METHOD

The Basic Idea: To illustrate the basic ideas of this method, the following nonlinear differential equation was considered He (2000):

$$A(u) - f(r) = 0, \qquad r \in \Omega, \qquad \textbf{(1)}$$

Subject to the boundary condition of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma, \qquad \textbf{(2)}$$

Where *A* is a general differential operator, *B* a boundary operator, *f(r)* a known analytical function and *Γ is* the boundary of the domain *Ω*. The operator *A* can be divided into two parts of *L* and *N*, where *L* is the linear part, while *N* is a nonlinear one eqn. (1) can therefore, be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \qquad r \in \Omega, \qquad \textbf{(3)}$$

The homotopy perturbation structure is shown as follows:

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \qquad \textbf{(4)}$$

Where:

$$v(r, p): \Omega \in [0,1] \to R \quad \textbf{(5)}$$

In Equation (4) $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. It can be assumed that the solution of Equation (4) can be written as power series in p as follows:

$$v = v_0 + pv_1 + p^2 v_2 + \ldots\ldots\ldots \quad \textbf{(6)}$$

And the best approximation for the solution is:

$$u = \lim_{p\to 1} v = v_0 + pv_1 + p^2 v_2 + \ldots\ldots\ldots \quad \textbf{(7)}$$

The series (7) is convergent for most cases. However, the convergence rate depends on the nonlinear operator *A(v)*. The following opinions are suggested by He (2000, 2006):

- The second derivative of *N(v)* with respect to *v* must be small because the parameter *p* may be relatively large, i.e. *p→1*.
- The norm of $L^{-1} \dfrac{\partial N}{\partial V}$ must be smaller than one so that the series converges.

IMPLEMENTATION OF THE METHOD

In order to illustrate the advantage of this method the following differential equations were considered:

(i) VAN DER POL EQUATION in the form of Aiyesimi and Adeniyi (2008):

$$\ddot{X} + n^2 X = \varepsilon\left(1 - X^2\right)\dot{X} \quad \textbf{(8)}$$

Subject to the conditions

$$X(0) = 0 \ and \ \dot{X}(0) = \tau_0$$

Equation (8) has wide application in the field of Science and Engineering. In order to solve (8) using HPM, a homotopy-perturbation method can be constructed as follows:

$$(1-p)\frac{d^2X}{dt^2} + p\left(\frac{d^2X}{dt^2} + n^2 X - \in\left(1 - X^2\right)\frac{dX}{dt}\right) = 0 \quad \textbf{(9)}$$

Substituting v from Equation (6) into (9) and equating the coefficient equal power of p terms, one has:

$$p^0: \quad \frac{d^2 v_0}{dt^2} = 0 \quad \textbf{(10)}$$

$$p^1: \quad \frac{d^2 v_1}{dt^2} + n^2 v_0 - \in\left(1 - v_0^{\ 2}\right)\frac{dv_0}{dt} = 0 \quad \textbf{(11)}$$

$$p^2: \quad \frac{d^2 v_2}{dt^2} + n^2 v_1 - \in\left[\left(1 - v_0^{\ 2}\right)\frac{dv_1}{dt} - 2v_0 v_1 \frac{dv_0}{dt}\right] = 0 \quad \textbf{(12)}$$

Solution of Equations (10) to (12) yields:

$$v_0 = \tau_0 t \quad \textbf{(13)}$$

$$v_1 = -\frac{1}{6}\tau_0 n^2 t^3 + \frac{1}{2} \in \tau_0 t^2 - \frac{1}{12} \in \tau_0^{\ 3} t^4 + \tau_0 t \quad \textbf{(14)}$$

$$v_2 = \frac{1}{84} \in^2 \tau_0^{\ 5} t^7 + \frac{1}{360} \in \tau_0^{\ 3} t^6 + \frac{1}{36} \in n^2 \tau_0^{\ 3} t^6 - \frac{7}{60} \in^2 \tau_0^{\ 3} t^5 + \frac{1}{120} n^3 \tau_0 t^5 - \frac{1}{24} \in n^2 \tau_0 t^4$$
$$\frac{1}{24} \in n \tau_0 t^4 - \frac{1}{4} \in \tau_0^{\ 3} t^4 + \frac{1}{6} \in^2 \tau_0 t^3 - \frac{1}{6} n \tau_0 t^3 + \frac{1}{2} \in \tau_0 t^2 + \tau_0 t \quad \textbf{(15)}$$

When $p \to 1$ the solution of Equation (8) becomes:

$$X(t) = \lim_{p \to 1} v = v_0 + pv_1 + p^2 v_2 \quad \textbf{(16)}$$

Hence,

$$X(t) = 3\tau_0 t - \frac{1}{6} n^2 \tau_0 t^3 + \in \tau_0 t^2 - \frac{1}{3} \in \tau_0{}^3 t^4 + \frac{1}{84} \in^2 \tau_0{}^5 t^7 + \frac{1}{360} n \in \tau_0{}^3 t^6 + \frac{1}{36} n^2 \in \tau_0{}^3 t^6$$
$$- \frac{7}{60} \in^2 \tau_0{}^3 t^5 + \frac{1}{120} n^3 \tau_0 t^5 - \frac{1}{24} \in n^2 \tau_0 t^4 - \frac{1}{24} \in n \tau_0 t^4 + \frac{1}{6} \in^2 \tau_0 t^3 - \frac{1}{6} n \tau_0 t^3 \quad \textbf{(17)}$$

(ii) EVOLUTION EQUATION in the form of Yan-ze and Krishnan (2006)

$$U_{tt} + \alpha U_{xx} + \beta U + \gamma U^3 = 0 \quad \textbf{(18)}$$

Subject to the conditions:

$$U(x,0) = 0 \quad and \quad U_t(x,0) = \sin(\lambda x)$$

In order to solve Equation (17) using HPM, a homotopy-perturbation method can be constructed as follows:

$$(1-p)\frac{\partial^2 U}{\partial t^2} + p\left(\frac{\partial^2 U}{\partial t^2} + \alpha \frac{\partial^2 U}{\partial x^2} + \beta U + \gamma U^3\right) = 0 \quad \textbf{(19)}$$

Substituting v from Equation (6) into (19) and equating the coefficient equal power of p terms, one has:

$$p^0: \quad \frac{\partial^2 v_0}{\partial t^2} = 0 \quad \textbf{(20)}$$

$$p^1: \quad \frac{\partial^2 v_1}{\partial t^2} + \alpha \frac{\partial^2 v_0}{\partial x^2} + \beta v_0 + \gamma v_0{}^3 = 0 \quad \textbf{(21)}$$

$$p^2: \quad \frac{\partial^2 v_2}{\partial t^2} + \alpha \frac{\partial^2 v_1}{\partial x^2} + \beta v_1 + 3\gamma v_0{}^2 v_1 = 0 \quad \textbf{(22)}$$

Solution of Equations (20) to (22) yields:

$$v_0 = t\sin(\lambda x) \quad \textbf{(23)}$$

$$v_1 = \frac{t}{60}\sin(\lambda x)\left[3\gamma t^4 \cos(\lambda x)^2 - 3\gamma t^4 + 10\alpha\lambda^2 t^2 + 10\beta t^2 + 60\right] \quad \textbf{(24)}$$

$$v_2 = \frac{t}{53760}\Big[7\gamma^2 t^8 \sin(5\lambda x) + 70\gamma^2 t^8 \sin(\lambda x) - 35\gamma^2 t^8 \sin(3\lambda x) + 304\alpha\gamma\lambda t^6 \sin(3\lambda x)$$
$$- 528\beta t^6 \sin(\lambda x) + 176\beta\gamma t^6 \sin(3\lambda x) - 528\alpha\gamma\lambda^2 t^6 \sin(\lambda x) + 448\alpha^2\lambda^4 t^4 \sin(\lambda x)$$
$$+ 896\beta\lambda^2\alpha t^4 \sin(\lambda x) + 448\beta^2 t^4 \sin(\lambda x) - 6048\gamma\lambda t^4 \sin(\lambda x) + 2016\gamma t^4 \sin(3\lambda x)$$
$$+ 8960\lambda^2\alpha t^2 \sin(\lambda x) + 896048\beta t^2 \sin(\lambda x) + 53760\sin(\lambda x)\Big] \quad \textbf{(25)}$$

When $p \to 1$ the solution of Equation (3.11) becomes:

$$U(x,t) = \lim_{p\to 1} v = v_0 + pv_1 + p^2 v_2$$

Hence

$$U(x,t) = \frac{t}{53760}\Big[7\gamma^2 t^8 \sin(5\lambda x) + 70\gamma^2 t^8 \sin(\lambda x) - 35\gamma^2 t^8 \sin(3\lambda x) + 304\alpha\gamma\lambda t^6 \sin(3\lambda x)$$
$$- 528\beta t^6 \sin(\lambda x) + 176\beta\gamma t^6 \sin(3\lambda x) - 528\alpha\gamma\lambda^2 t^6 \sin(\lambda x) + 448\alpha^2\lambda^4 t^4 \sin(\lambda x)$$
$$+ 896\beta\lambda^2\alpha t^4 \sin(\lambda x) + 448\beta^2 t^4 \sin(\lambda x) - 8736\gamma t^4 \sin(\lambda x) + 2016\gamma t^4 \sin(3\lambda x)$$
$$+ 17920\lambda^2\alpha t^2 \sin(\lambda x) + 17920\beta t^2 \sin(\lambda x) + 161280\sin(\lambda x) + 2688\gamma t^4 \sin(\lambda x)\cos(\lambda x)^2\Big] \quad \textbf{(26)}$$

ANALYSIS OF THE RESULTS

The behavior of $X(t)$ and $U(x,t)$ obtained by the Homotopy Perturbation Method is shown in the following figures:

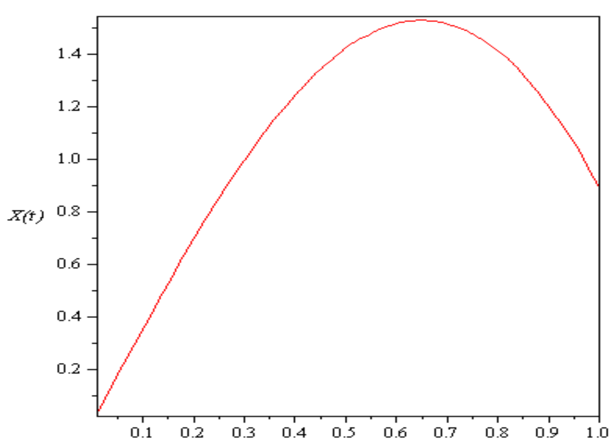


Figure 1: The Behavior of $X(t)$ obtained by HPM with $\in$=0.01.

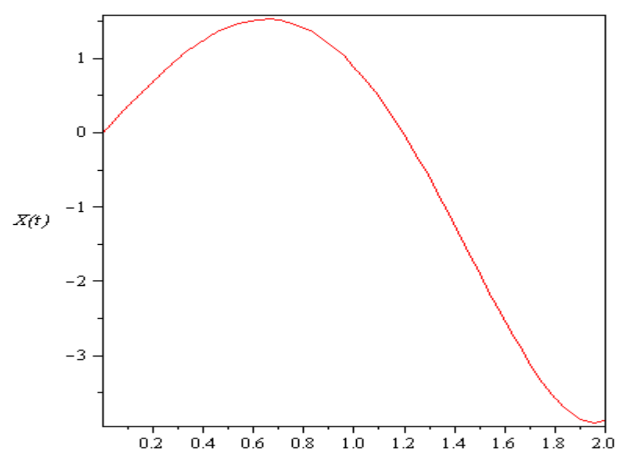
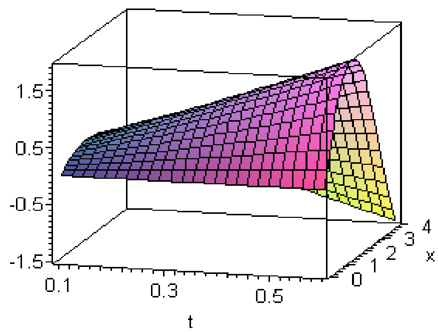
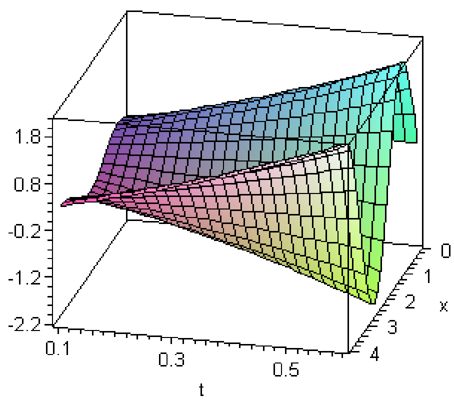


Figure 2: The Behavior of $X(t)$ obtained by HPM with $\in$=0.1



(a)



(b)

Figure 3: The Behavior of *U(x,t)* Obtained by HPM with α=1,β=1,γ=1 at (a) λ=1 (b) λ=2.

CONCLUSION

In this study, the HPM for solving the Van der pol and Evolution equations were successfully developed. It is observed from the results that HPM is a very powerful and efficient technique for finding solutions for wide classes of linear and non-linear problems in the form of analytical expressions, the solution to the equations tends to converge rapidly.

The behavior of X(t) over time t with fixed value ∈ are shown in Figure 1 and Figure 2. The efficiency of HPM at some fixed values for U(x,t) is shown in Figure 3 (a)-(b). It is observed that this method avoids restrictive assumptions or transformations and physically unrealistic assumptions.

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