On Exact Finite-Difference Scheme for Numerical Solution of Initial Value Problems in Ordinary Differential Equations.

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ABSTRACT

This paper presents a numerical method called the Exact Finite-Difference Scheme for the solution of Ordinary Differential Equations of first-order. The need for exact finite difference scheme came up due to some shortcomings of the standard methods; in which the qualitative properties of the exact solution are not usually transferred to the numerical solution. These shortcomings create problems which may affect the stability property of the standard approach. The exact finite-difference scheme has the property that their solutions do not have numerical instabilities.

(Keywords: exact finite-difference scheme, non-standard finite difference scheme, initial value problems, ODEs, models)

INTRODUCTION

It is a known and documented fact that a given linear or non-linear equation does not have a complete solution that can be expressed in terms of a finite number of elementary functions. It is also a known fact that one of the ways to solve such problem is to seek an approximate solution by means of various perturbation methods (Rose, 1964 and Humi and Miller, 1989). It must be stated here that the above procedure will only hold for limited ranges of the system parameters and the independent variable (Mickens, 1994). As reported in Mickens (1994), for arbitrary values of the system parameters at the present time, only numerical integration technique can provide accurate solutions to the original differential equation.

For any numerical method to be convergent, it has to be a sufficiently accurate representation of the differential system (Lambert, 1991). It has been observed that the exact finite difference scheme (which is a special case of non-standard finite-difference scheme) is one that does not exhibit numerical instability. Throughout this work, we shall consider the general first-order differential equation:

$$\frac{dy}{dt} = f(y, x, \lambda), y(t_0) = y_0 \quad \textbf{(1)}$$

where $f(y, x, \lambda)$ is such that Equation (1) has a unique solution over the interval, $0 \le t \le T$ and for λ in the interval $\lambda_1 \le \lambda \le \lambda_2$. Equation (1) occurs in physical and biological sciences, management sciences and engineering. In fact, the importance of solving equations of the form (1) cannot be over emphasized.

For dynamical systems of interest, in general, $T = \infty$, i.e. the solution exist for all time. This solution can be written as:

$$y(t) = \phi(\lambda, y_0, t_0, t) \quad \textbf{(2)}$$

with

$$\phi(\lambda, y_0, t_0, t) = y_0 \quad \textbf{(3)}$$

A discrete model of Equation (1) can be written as:

$$y_{n+1} = g(\lambda, h, y_n, t_n),\ t_n = hn \quad \textbf{(4)}$$

The solution to equation (4) can be expressed as,

$$y_n = \eta(\lambda, h, y_0, t_0, t_n) \quad \textbf{(5)}$$

with

$$\eta(\lambda, h, y_0, t_0, t_n) = y_0 \quad \textbf{(6)}$$

Definition 1 (Mickens 1994)

Equations (1) and (4) are said to have the same general solution if and only if,

$$y_n = y(t_n) \qquad \textbf{(7)}$$

for arbitrary values of h.

Definition 2 (Mickens 1994)

An exact finite difference scheme is one for which the solution to the difference equation have the same general solution as the associated differential equation. These definitions lead to the following result.

Theorem 1 (Mickens 1994)

The differential equation (1) has an exact finite-difference scheme given by the expression:

$$y_{n+1} = \phi[\lambda, y_n, t_n, t_{n+1}] \qquad \textbf{(8)}$$

where ϕ is that of Equation (2).

Proof

The group property of the solution to (1) gives,

$$y(t+h) = \phi[\lambda, y(t), t, t+h] \qquad \textbf{(9)}$$

If we now make the identifications,

$$t \to t_n,\ y(t) \to y_n \qquad \textbf{(10)}$$

then (9) becomes,

$$y_{n+1} = \phi(\lambda, y_n, t_n, t_{n+1}) \qquad \textbf{(11)}$$

This is the required ordinary difference equation that has the same general solution as (1).

Remarks

i) If all solutions of (1) exist for all time, i.e. $T = \infty$, Then (1) holds for all t and h. Otherwise, the relation is assumed to hold whenever the right-side of (9) is well defined.

ii) The theorem is only an existence theorem. It basically says that if an ordinary differential equation has a solution, then an exact finite-difference scheme exists.

iii) A major implication of the theorem is that the solution of the difference equation is exactly equal to the solution of the ordinary differential equation on the computation grid for fixed, but arbitrary step-size h.

iv) The theorem can be easily generalized to systems of coupled, first-order ordinary differential equations.

The discovery of exact discrete models for ordinary differential equations is of great importance, primarily because it allows us to gain insights into the better construction of finite-difference schemes. They also provide computational investigator with useful benchmarks for comparison with the standard procedures.

THE GENERAL THEORY OF NON-STANDARD METHODS

Consider the differential equation:

$$\frac{dy}{dx} = f(y, \lambda) \qquad \textbf{(12)}$$

Where λ is an n-parameter vector. Equation (12) can be written in the form:

$$\frac{y_{n+1} - y_n}{\phi(h, \lambda)} = F(y_n, y_{n+1}, \lambda, h) \qquad \textbf{(13)}$$

If

$$\frac{dy}{dx} \to \frac{y_{n+1} - y_n}{h} \qquad \textbf{(14)}$$

Then Equation (13) is a generalization of Equation (14). In this case,

$$\frac{dy}{dx} \rightarrow \frac{y_{n+1} - y_n}{\phi(\lambda, h)} \quad \textbf{(15)}$$

Where $\phi(\lambda, h)$, the denominator function has the property that:

$$\phi(\lambda, h) = h + o(h^2)$$
$$\lambda = fixed,\ h \rightarrow 0 \quad \textbf{(16)}$$

The above formulation is based on the traditional definition of the derivative which is of the form

$$\frac{dy}{dx} = \underset{h \rightarrow 0}{Lim} \frac{y(x + \psi_1(h)) - y(x)}{\psi_2(h)} \quad \textbf{(17)}$$

Where $\psi_i(h) = h + o(h^2),\ h \rightarrow 0, i = 1, 2$.

Example of functions $\psi(h)$ that satisfy condition (17) above are:

$$\psi(h) = \begin{cases} h \\ \sin(h) \\ e^h - 1 \\ 1 - e^{-h} \\ \dfrac{1 - e^{-\lambda h}}{\lambda} \\ etc \end{cases} \quad \textbf{(18)}$$

The values of $\psi_i, i = 1, 2, ..., n$ depends on the differential equation under consideration. It must be stated here that if $h \rightarrow 0$, then

$$\frac{dy}{dx} = \underset{h \rightarrow 0}{Lim} \frac{y(x + \psi_1(h)) - y(x)}{\psi_2(h)} = \underset{h \rightarrow 0}{Lim} \frac{y(x + h) - y(x)}{h} \quad \textbf{(19)}$$

NON-STANDARD FINITE DIFFERENCE MODELLING RULES

The general form of non-standard method can be written as:

$$y_{n+1} = F(h, y_n) \quad \textbf{(20)}$$

Non-standard finite difference schemes were developed using a collection of rules set by Mickens as follows:

<u>Rules 1 (Mickens 1994)</u>

The order of the discrete derivative must be exactly equal to the order of the corresponding derivatives of the differential equation. If this rule is violated, this can lead to numerical instability in the form of oscillations which may be bounded or unbounded. The mathematical reason for the above occurrence is that discrete equations have large class of solutions than differential equations.

As an illustration, let us consider the following first order differential equation:

$$\frac{dy}{dx} = -y \quad \textbf{(21)}$$

If we model (21) by a central difference scheme of the form:

$$\frac{y_{n+1} - y_{n-1}}{2h} = -y_n \quad \textbf{(22)}$$

It will be discovered that this modeling has extra solution that is strange because Equation (22) is of second order while (21) is of first order, thus the principle of uniqueness is violated and this leads to the existence of numerical instability.

<u>Rule 2 (Mickens 1994)</u>

Denominator function for the discrete derivatives must be expressed in terms of more complicated function of the step-sizes than those conventionally used. This rule allows the introduction of complex analytic function of h in the denominator. For instance, consider,

$$\frac{dy}{dx} = y(1 - y) \quad \textbf{(23)}$$

This is in form of a logistic equation. If the denominator function D is given by,

$$D_1 = e^h - 1 \quad \textbf{(24)}$$

then substituting Equation (24) in Equation (15) gives,

$$\frac{y_{n+1}-y_n}{e^h-1}=y_n(1-y_{n+1}) \quad \textbf{(25)}$$

It must be stated here that the selection of an appropriate denominator is an 'art' (Mickens, 1999). We must examine the differential equation for which the exact schemes are known. Close examination of differential equation for which exact schemes are known, shows that the denominator function generally are functions that are related to particular solution or properties of the general solution to the difference equation.

Rule 3 (Mickens 1994)

The non-linear terms must in general be modeled (approximated) non-locally on the computational grid or lattice in many different ways, for instance, in Equation (25), it is assumed that $y^2 \cong y_n y_{n+1}$. The non-linear terms y^2, y^3 can be modeled as follows:

$$y^2 \cong y_n y_{n+1} \quad \textbf{(26)}$$

$$y^2 \cong y_n(\frac{y_{n+1}+y_n}{2}) \quad \textbf{(27)}$$

$$y^3 \cong y_n^2 y_{n+1} \quad \textbf{(28)}$$

$$y^3 \cong y_n^2(\frac{y_{n+1}+y_n}{2}) \quad \textbf{(29)}$$

The particular form selected from Equations (26) to (29) depends on the full discrete model.

Rule 4 (Mickens 1994)

Special solutions of the differential equations should also be accompanied by special discrete solutions of the finite-difference models.

Rule 5 (Mickens 1994)

The finite-difference equation should not have solutions that do not correspond exactly to the solution of the differential equations. The non-standard methods shall be applied to some problems as shown below:

Example 1

The non-standard finite-difference scheme for the solution of

$$y' = y^2, y(0)=1 \quad \textbf{(30)}$$

is presented using the following approximations:

$$y^2 \cong y_n y_{n+1} \quad \textbf{(31)}$$

$$y^2 \cong y_n \frac{y_{n-1}+y_{n+1}}{2} \quad \textbf{(32)}$$

Using Equation (31), Equation (30) can be written as:

$$\frac{y_{n+1}-y_n}{\phi(h)}=y_n y_{n+1} \quad \textbf{(33)}$$

which is in the form of Equation (15).

$$y_{n+1}-y_n=\phi(h)y_n y_{n+1} \quad \textbf{(34)}$$

$$y_{n+1}(1-\phi(h)y_n)=y_n \quad \textbf{(35)}$$

This can be written in a compact form as:

$$y_{n+1}=\frac{y_n}{1-\phi(h)y_n} \quad \textbf{(36)}$$

Equation (36) is of the form (20) which is in non-standard form. Now, using Equation (32), Equation (30) can be written as:

$$\frac{y_{n+1}-y_n}{\phi(h)}=y_n(\frac{y_{n-1}+y_{n+1}}{2}) \quad \textbf{(37)}$$

$$\Rightarrow \frac{y_{n+1}-y_n}{\phi(h)} = \frac{y_n y_{n-1} + y_n y_{n+1}}{2} \quad \textbf{(38)}$$

That is,

$$2y_{n+1} - 2y_n = \phi(h) y_n y_{n-1} + \phi(h) y_n y_{n+1} \quad \textbf{(39)}$$

Equation (39) simplifies to:

$$y_{n+1} = \frac{(2+\phi(h)y_{n-1})y_n}{2-\phi(h)y_n} \quad \textbf{(40)}$$

Equation (40) can be written in the form:

$$y_{n+2} = \frac{(2+\phi(h)y_n)y_{n+1}}{2-\phi(h)y_{n+1}} \quad \textbf{(41)}$$

Equation (41) is also of the form (20).

Example 2

Consider the differential equation:

$$\frac{dy}{dx} = y^2(y^2-1) \quad \textbf{(42)}$$

Non-standard difference scheme is constructed for the solution of the Equation (42) by approximating $y^2 \cong y_n y_{n+1}$. Equation (42) becomes:

$$\frac{y_{n+1}-y_n}{\phi(h)} = y_n y_{n+1}(y_n y_{n+1} - 1) \quad \textbf{(43)}$$

$$y_{n+1} - y_n = \phi(h) y_n y_{n+1}(y_n y_{n+1} - 1) \quad \textbf{(44)}$$

$$y_{n+1} - y_n = \phi(h) y_n^2 y_{n+1}^2 - \phi(h) y_n y_{n+1} \quad \textbf{(45)}$$

$$y_{n+1} + \phi(h) y_n y_{n+1} = y_n + \phi(h) y_n^2 y_{n+1}^2 \quad \textbf{(46)}$$

That is,

$$y_{n+1}(1+\phi(h)y_n) = y_n + \phi(h) y_n^2 y_{n+1}^2 \quad \textbf{(47)}$$

which leads to,

$$y_{n+1} = \frac{y_n + \phi(h) y_n^2 y_{n+1}^2}{1+\phi(h)y_n} \quad \textbf{(48)}$$

<u>Theorem 2 (Anguelov and Lubuma 2003)</u>

The finite difference scheme (20) is stable with respect to monotone dependence on initial value, if:

$$\frac{\partial F(h,y)}{\partial y} \geq 0, y \in R, h > 0 \quad \textbf{(49)}$$

<u>Theorem 3 (Anguelov and Lubuma 2003)</u>

The non-standard scheme (36) is stable with respect to monotone dependence on initial value and monotone of solution and therefore is elementary stable.

<u>Proof</u>

Consider the scheme (36):

$$F(h, y_n) = y_{n+1} = \frac{y_n}{1-\phi(h)y_n} \quad \textbf{(50)}$$

Here,

$$F(h,y) = \frac{y}{1-\phi(h)y} \quad \textbf{(51)}$$

$$\frac{\partial F}{\partial y} = \frac{[1-\phi(h)y]\frac{\partial(y)}{\partial y} - y\frac{\partial}{\partial y}[1-\phi(h)y]}{[1-\phi(h)y]^2} \quad \textbf{(52)}$$

$$= \frac{1-\phi(h)y+\phi(h)y}{[1-\phi(h)y]^2} \quad \textbf{(53)}$$

$$= \frac{1}{[1-\phi(h)y]^2} \geq 0 \quad \textbf{(54)}$$

$$\Rightarrow \frac{\partial F(h,y)}{\partial y} \geq 0 \quad \textbf{(55)}$$

Here, since $\phi(h)$ is positive and h>0, this implies that the scheme (36) is stable with respect to monotone dependence on initial value.

DERIVATION OF THE EXACT FINITE-DIFFERENCE SCHEME

In this section, the exact-finite difference scheme capable of producing an exact solution to problems in form of Equation (1) shall be derived. The discovery of exact discrete models for particular ordinary differential equations is of great importance, primarily because it allows us to gain insights into the better construction of finite-difference schemes. They also provide the computational investigator with useful benchmarks for comparison with the standard procedures. Above all, a major advantage of having an exact difference equations model for ordinary differential equation is that questions related to the usual consideration of consistency, stability, and convergence need not arise (Mickens 1994).

Consider the equation of the form (1). If we assume that the exact (theoretical) solution of (1), at point $x = x_n$ denoted by $y(x_n)$ has the same general solution with numerical solution (i.e. the new exact finite scheme), at point $x = x_n$ denoted by y_n, then;

$$y_n = y(x_n) \quad \textbf{(56)}$$

This implies that at point $x = x_{n+1}$,

$$y_{n+1} = y(x_{n+1}) \quad \textbf{(57)}$$

The following determinant gives the required difference equation:

$$\begin{vmatrix} y_n & y(x_n) \\ y_{n+1} & y(x_{n+1}) \end{vmatrix} = 0 \quad \textbf{(58)}$$

It is obvious from equation (58) that,

$$y_{n+1} = \frac{y(x_{n+1})}{y(x_n)} y_n \quad \textbf{(59)}$$

This is the exact finite scheme capable of solving any equation of the form (1). It is important to note that the scheme (59) is of the form (20).

One of the shortcomings of an exact finite difference scheme is that it is necessary that we must know the theoretical solution before we can construct the method. The advantage of exact finite difference scheme is that it produces exact solution to the differential equations under consideration. Another advantage is that, questions related to the usual consideration of consistency, stability and convergence need not arise (Mickens 1994).

<u>Construction of Exact-Finite Difference Schemes</u>

We shall now consider how the exact finite-difference scheme is being constructed using the following test problems.

<u>Problem 1</u>

Consider the Initial Value Problem,

$$y' = 4x - 2xy, \;\; y(0) = 3 \quad \textbf{(60)}$$

with the theoretical (exact) solution,

$$y(x) = 2 + e^{-x^2} \quad \textbf{(61)}$$

At the point $x = x_{n+1}$, we have,

$$y_n = 2 + e^{-x_n^2} \quad \textbf{(62)}$$

and at the point $x = x_{n+1}$, we have,

$$y_{n+1} = 2 + e^{-x_{n+1}^2} \quad \textbf{(63)}$$

The following determinant gives the required difference equation,

$$\begin{vmatrix} y_n & 2 + e^{-x_n^2} \\ y_{n+1} & 2 + e^{-x_{n+1}^2} \end{vmatrix} = 0 \quad \textbf{(64)}$$

$$\Rightarrow y_{n+1} = y_n \frac{(2+e^{-x_{n+1}^2})}{(2+e^{-x_n^2})} \quad \textbf{(65)}$$

where $x_n = nh \ and \ x_{n+1} = (n+1)h$. The exact finite-difference scheme (65) is capable of solving Equation (60).

Problem 2

Consider the Initial Value Problem

$$y^{'} = y^2,\ y(0) = 1 \quad \textbf{(66)}$$

with the theoretical (exact) solution,

$$y(x) = \frac{1}{1-x} \quad \textbf{(67)}$$

The following determinant gives the required difference equation,

$$\begin{vmatrix} y_n & \frac{1}{1-x_n} \\ y_{n+1} & \frac{1}{1-x_{n+1}} \end{vmatrix} = 0 \quad \textbf{(68)}$$

$$\Rightarrow y_{n+1} = y_n \frac{(1-nh)}{[1-(n+1)h]} \quad \textbf{(69)}$$

Equation (69) is the exact finite-difference scheme for solving Equation (66).

Problem 3

Consider the Initial Value Problem,

$$y^{'} = -0.0026y,\ y(0) = 100 \quad \textbf{(70)}$$

with the theoretical (exact) solution,

$$y(x) = 100e^{-0.0026x} \quad \textbf{(71)}$$

The determinant below gives the required difference equation,

$$\begin{vmatrix} y_n & 100e^{-0.0026x_n} \\ y_{n+1} & 100e^{-0.0026x_{n+1}} \end{vmatrix} = 0 \quad \textbf{(72)}$$

$$\Rightarrow y_{n+1} = y_n e^{-0.0026h} \quad \textbf{(73)}$$

This is the exact finite-difference scheme for Equation (70). For the applications of the constructed schemes above, see Ibijola, E. A. and Sunday, J. (*Aust. J. of Basic & Appl. Sci., 4(4):624-632, 2010*).

CONCLUSION

From the presentation above, we conclude that the Exact Finite-Difference Scheme is computationally reliable and efficient. This is because it performs well on initial value problems of ordinary differential equations, in fact the numerical solution does not exhibit numerical instability.

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