

A Six Step Block Method for Solution of Fourth Order Ordinary Differential Equations.

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ABSTRACT

A linear multistep method for solving first order initial value problems of ordinary differential equations is presented in this paper. The approach of collocation approximation is adopted in the derivation of the scheme and then the scheme is applied as simultaneous integrator to first order initial value problems of ordinary differential equations. This implementation strategy is more accurate and efficient than those given when the same scheme is applied over overlapping intervals in predictor-corrector mode. Furthermore, the new block method possesses the desirable feature of the Runge-Kutta method of being self-starting and eliminates the use of the predictor-corrector method. Experimental results confirm the superiority of the new scheme over the existing methods.

(Keywords: linear multistep methods, LMMs, p-stability, zero-stability)

INTRODUCTION

Recent research in this area includes Awoyemi and Idowu (2005), a class of hybrid collocation method for general third order of ordinary differential equations. Awoyemi (2003) developed a p-stable linear multistep method for general third order initial value problems of ordinary differential equations but the implementation strategy is in predictor-corrector mode. Like other linear multistep methods and other standard methods, these are usually applied to the initial value problems as a single formula but the drawbacks of the methods are well known. Firstly, they are not self-starting; secondly, they advance the numerical integration of the ordinary differential equations one-step at a time, which leads to overlapping of the piecewise polynomials solution model (Yusuph, 2004). Moreover, the overlapping creates a

disadvantage because the numerical model fails to represent the solution uniquely elsewhere than the mesh-points.

For boundary value problems, this is an important criticism of the linear multistep methods in favor of the finite element methods (Jennings, 1997).

Jator (2007) presented a class of initial value methods for the direct solution of second order initial value problems. In his work, LMMs with continuous coefficients were obtained and applied as simultaneous numerical integrators to $y' = f(x, y, y')$.

The implementation strategy is more efficient than those given in Awoyemi (1999) which are applied over overlapping intervals in predictor-corrector mode. Vigo-Aguiar and Ramos (2006) constructed a variable step-size implementation of multistep methods for $y'' = f(x, y, y')$.

Yusuph and Onumanyi (2002) derived new multiple FDMS through multistep collocation for $y' = f(x, y)$.

This paper therefore proposes a 6-step block scheme for the solution of special fourth order ordinary differential equations which eliminates the use of predictors by providing sufficiently accurate simultaneous difference equations from a single continuous formula and its derivative.

The Multistep Collocation Method

In the spirit of Onumayi et al. (1994) and Yahaya (2004) we consider the construction of multistep collocation method of constant step size h , though h can be variable and give continuous expression for the coefficient. The values of K

and M are arbitrary except for collocation at the mesh points where $0 < m \leq k + 1$

Let y_{n+j} be approximation to y_{n+j} where

$$y_{n+j} = Y_{n+j}, j = 0, \dots, K-1$$

Then a K-step multistep collocation method is constructed as follows. We find a polynomial $y(x)$ of degree $p = t + m - 1, t > 0, m > 0$ and such that it satisfies the conditions.

$$y(x_{n+j}) = y_{n+j}, j = 0, \dots, k \quad (1)$$

$$y''(x_{n+j}) = f(x_{n+j}), j = 0, \dots, m-1 \quad (2)$$

where x_1, \dots, x_{m-1} are free collocation points, we then take as an approximation to $y_{n+k}, Y_{n+k} = Y_{n+k}$.

Let

$$y(x) = \sum_{j=1}^{k-1} \alpha_j y_{n+j} + h^2 \sum_{j=1}^{m-1} \beta_j f(x_{n+j}), y(x_{n+j}) \quad (3)$$

where α_j and β_j are assumed polynomial of the form,

$$\alpha_j(x) = \sum_{i=0}^{m+t-1} a_j x^i; h^2 \beta_j(x) = \sum_{i=0}^{m+t-1} h^2 \beta_j x^i \quad (4)$$

and the collocation point x_{j+1} in (3) belong to the extended set,

$$Q = \{x_1, \dots, x_{n+k}, x_{n+k+1}, \dots, x_{n+k}\} \quad (5)$$

From the interpolation conditions (1) and the expression for $y(x)$ in (3) the following conditions are imposed on $\alpha_j(x)$ and $\beta_j(x)$,

$$\alpha_j(x_{m+i}) = \delta_{ij}, j = 0, \dots, t-1, i = 0, \dots, t-1$$

$$h^2 \beta_j(x_{m+i}) = 0, j = 0, \dots, m-1, i = 0, \dots, t-1 \quad (6)$$

and

$$\alpha_j^{(i)}(x_{n+i}) = 0, j = 0, \dots, t-1, i = 0, \dots, m-1 \quad (7)$$

$$h^2 \beta_j(x_{n+i}) = \delta_{ij}, j = 0, \dots, m-1, i = 0, \dots, m-1$$

Next we write (6) – (7) in a matrix equation of the form:

$$DC = I \quad (8)$$

where I is the identity matrix of dimension $(m+t)$. The matrices D and C are both of dimensions $(m+t) \times (m+t)$. It follows from (8) that the columns of $C = D^{-1}$ give the continuous coefficient $\alpha_j(x)$ and $\beta_j(x)$.

Derivation of the Present Method

We propose an approximate solution to (1) in the form:

$$y_p(x) = \sum_{j=0}^{m+t-1} a_j x^j, i = 0, \dots, m+t-1 \quad (9)$$

$$y_p''(x) = \sum_{j=0}^{m+t-1} i(i-1)(i-2)(i-3) a_j x^{i-4}, i = 2, 3, \dots, m+t-3 \quad (10)$$

with $m = 4, t = 4$ and $p = m + t - 1$ also

$\alpha_j, \beta_j, j = 0, 1, \dots, m+t-1$ are the parameters to be determined. Where p , is the degree of the polynomial interpolant of our choice Specifically, we collocate Equation (10) at $x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ and interpolate Equation (9) at $x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ using the method described above in this paper, we obtain a continuous form for the solution $y(x) = VC^T p(x)$ from the system of the equation in the matrix form below:

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{pmatrix} \begin{pmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 \\ 0 & 0 & 0 & 0 & 24 & 120x_n & 360x_n^2 & 840x_n^3 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x_{n+2}^2 & 840x_{n+2}^3 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+5} & 360x_{n+5}^2 & 840x_{n+5}^3 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+6} & 360x_{n+6}^2 & 840x_{n+6}^3 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ f_n \\ f_{n+2} \\ f_{n+5} \\ f_{n+6} \end{pmatrix} \quad (11)$$

Matrix D in Equation (11), which when solved either by matrix inversion techniques or Gaussian elimination method to obtain the values of the

Parameters $\alpha_j, j=0,1,\dots,m+t-1$ and then substituting them into equation (9) give a scheme expressed in the form.

$$y_k \approx \sum_{j=0}^{k-1} \alpha_j y_{n+j} + \sum_{j=0}^{k-2} \beta_j f_{n+j} \quad (12)$$

If we now let $k=6$, after some manipulations we obtain a continuous form of solution (13).

$$\begin{aligned}
 y \approx & \left[\frac{1}{6h^3} \left(\left(-x_n \right)^3 + 9 \left(-x_n \right)^2 h - 26 \left(-x_n \right) h^2 + 24 h^3 \right) \right] y_{n+1} \\
 & \left[\frac{1}{2h^3} \left(\left(-x_n \right)^3 - 8 \left(-x_n \right)^2 h + 19 \left(-x_n \right) h^2 - 12 h^3 \right) \right] y_{n+2} \\
 & \left[\frac{1}{2h^3} \left(\left(-x_n \right)^3 + 7 \left(-x_n \right)^2 h - 14 \left(-x_n \right) h^2 + 8 h^3 \right) \right] y_{n+3} \\
 & \left[\frac{1}{2h^3} \left(\left(-x_n \right)^3 - 6 \left(-x_n \right)^2 h + 11 \left(-x_n \right) h^2 - 6 h^3 \right) \right] y_{n+4} \\
 & + \left[\frac{3 \left(-x_n \right)^3 + 91 \left(-x_n \right)^2 h - 1092 \left(-x_n \right) h^2 + 6300 \left(-x_n \right) h^3 - 18767 \left(-x_n \right) h^4 + 28889 \left(-x_n \right) h^5 - 21298 \left(-x_n \right) h^6 + 5880 h^7}{151200 h^3} \right] f_n \\
 & + \left[\frac{1 \left(-x_n \right)^3 - 77 \left(-x_n \right)^2 h + 630 \left(-x_n \right) h^2 - 19103 \left(-x_n \right) h^4 + 78197 \left(-x_n \right) h^5 - 115930 \left(-x_n \right) h^6 + 56280 h^7}{60480 h^3} \right] f_{n+2} \\
 & + \left[\frac{3 \left(-x_n \right)^3 + 56 \left(-x_n \right)^2 h - 252 \left(-x_n \right) h^2 + 1883 \left(-x_n \right) h^4 - 2576 \left(-x_n \right) h^5 - 788 \left(-x_n \right) h^6 + 1680 h^7}{37800 h^3} \right] f_{n+5} \\
 & + \left[\frac{1 \left(-x_n \right)^3 - 49 \left(-x_n \right)^2 h + 210 \left(-x_n \right) h^2 - 1609 \left(-x_n \right) h^4 + 2569 \left(-x_n \right) h^5 - 290 \left(-x_n \right) h^6 + 840 h^7}{60480 h^3} \right] f_{n+6}
 \end{aligned} \quad (13)$$

The continuous scheme (13) was evaluated at some selected points it yielded the following discrete schemes:

Evaluating (13) at $x=x_{n+6}, x=x_{n+5}, x=x_n$, yield respectively three integrator below.

$$y_{n+6} + 4y_{n+1} - 15y_{n+2} + 20y_{n+3} - 10y_{n+4} = \frac{h^4}{360} \left[134f_n + 1165f_{n+2} + 1184f_{n+5} - 415f_{n+6} \right]$$

with order $p=4$ i.e. $C_{p+4}=3.006944$

$$y_{n+5} + y_{n+1} - 4y_{n+2} + 6y_{n+3} - 4y_{n+4} = \frac{h^4}{90} \left[7f_n + 65f_{n+2} - 52f_{n+5} - 20f_{n+6} \right]$$

with order $p=4$ i.e. $C_{p+4}=0.6097222$

$$y_{n+4} - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_n = \frac{h^4}{360} [4f_n + 335f_{n+2} + 16f_{n+5} - 5f_{n+6}]$$

with order p=4 i.e. $C_{p+4} = -0.0152778$ (14)

Improving the block method by considering additional equation arising from the first, second, and third derivative functions.

Taking the first derivative of equation (13), thereafter, evaluate the resulting continuous polynomial solution at $x=x_0$, yields:

$$-\frac{13}{3}y_{n+1} + \frac{19}{2}y_{n+2} - 7y_{n+3} + \frac{11}{6} - hz_0 = \frac{h^4}{151200} [1298f_n + 289825f_{n+2} + 3152f_{n+5} + 725f_{n+6}]$$
(15)

with order p=4 i.e. $C_{p+4} = -27012.5$.

Taking the second derivative of Equation (13), thereafter, evaluate the resulting continuous polynomial solution at $x=x_0$, yields:

$$-3y_{n+1} + 8y_{n+2} - 7y_{n+3} + 2y_{n+4} + h^2z_0 = \frac{h^4}{21600} [254f_n + 55855f_{n+2} + 3152f_{n+5} + 725f_{n+6}]$$
(16)

with order p=4 i.e. $C_{p+4} = 12951.071$.

Taking the third derivative of Equation (13), thereafter, evaluate the resulting continuous polynomial solution at $x=x_0$, yields:

$$-y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4} - h^3z_0 = \frac{h^4}{7200} [362f_n + 13645f_{n+2} - 2152f_{n+5} + 1145f_{n+6}]$$
(17)

with order p=4 i.e. $C_{p+4} = 6560$

Equation (14), (15), (16), and (17) constitute the member of a zero-stable block integrators of order $(4,4,4,4,4,4)^T$ with $C_8 = \left(\frac{433}{144}, \frac{439}{720}, -\frac{11}{720}, -\frac{54025}{2}, \frac{191315}{14}, 6560 \right)$ the application of the block integrators with $n = 0$ give the accurate values of y_1, y_2, y_3, y_4, y_5 along with y_6 as shown in Table 1.

To start the IVP integration on the sub interval $[x_0, x_6]$ we combine (14), (15), (16), and (17), when $n = 0$ (i.e., the 1-block 6-point method as given in Equation (18)). Thus produces simultaneously values for y_1, y_2, y_3, y_4, y_5 along with y_6 without recourse to any predictor

Stability Analysis

Recall that it is a desirable property for a numerical integrator to produce solutions that behave similar to the theoretical solution to a problem at all times. Thus several definitions, which call for the method to possess some "adequate" region of absolute stability, can be found in several literatures. See Lambert (1973), Fatunla (1992; 1994), etc., following Fatunla (1992; 1994), the six integrator proposed in this report in Equations (14), (15), (16), and (17), are put in the matrix-equation form and for easy analysis the result was normalized to obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h^4 \left\{ \begin{pmatrix} -23803 & -761 & -941 & -341 & 107 \\ 10080 & 2520 & 5040 & 5040 & 10080 \\ -9707 & -37 & 136 & -101 & 8 \\ 2520 & 63 & 315 & 630 & 315 \\ -3523 & -9 & 87 & -9 & 9 \\ 672 & 140 & 112 & 35 & 224 \\ -8363 & 176 & 608 & -16 & 16 \\ 1260 & 315 & 315 & 63 & 315 \\ -16243 & 625 & 3125 & 625 & 275 \\ 2016 & 504 & 1008 & 1008 & 2016 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1231 \\ 0 & 0 & 0 & 5040 \\ 0 & 0 & 0 & 71 \\ 0 & 0 & 0 & 126 \\ 0 & 0 & 0 & 123 \\ 0 & 0 & 0 & 140 \\ 0 & 0 & 0 & 376 \\ 0 & 0 & 0 & 315 \\ 0 & 0 & 0 & 152 \\ 0 & 0 & 0 & 1008 \end{pmatrix} \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \right\} \quad (18)$$

The first characteristic polynomial of the proposed 1-block 6- step method is

$$\begin{aligned}
P(R) &= \det \left[RA^6 - A^6 \right] \\
P(R) &= \det \left[R \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] \\
&= \det \begin{pmatrix} R & 0 & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & 0 & -1 \\ 0 & 0 & R & 0 & 0 & -1 \\ 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & R & 1 \\ 0 & 0 & 0 & 0 & 0 & R-1 \end{pmatrix} \\
&= R^5 (R-1)
\end{aligned} \quad (19)$$

$P(R) = R^5 (R-1)$ This implies, $R_1 = R_2 = R_3 = R_4 = R_5 = 0$ or $R_6 = 1$

From the equation (18), the 1 block 6-point is zero stable and is also consistent as its order $(4,4,4,4,4,4)^T > 1$, thus, it is convergent, following Henrici (1962)

Numerical Experiment

This section deals with the implementation of the algorithm proposed for fourth order initial value problems. Consider the initial value problem:

$$y^{(4)} = x,$$

$$0 \leq x \leq 1.0$$

$$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0$$

$$h = 0.1$$

exact solution :

$$y(x) = \frac{x^5}{120} + x$$

Table 2: Accuracy of 6-Step Block Method of Order 6, H=0.1.

Theoretical/Approximate Solutions, h=0.1			
X	Exact solution y(x)	6-step Block Method y-computed	Errors
0.1	0.100000083	0.1000000837	7.000000024E-10
0.2	0.200002667	0.2000026679	8.999999912E-10
0.3	0.30002025	0.3000202526	2.599999993E-09
0.4	0.400085333	0.4000853381	5.100000033E-09
0.5	0.500260417	0.5002604248	7.799999979E-09
0.6	0.600648	0.6006480118	1.180000009E-08
0.7	0.701400583	0.7014005954	1.240000003E-08
0.8	0.802730667	0.8027306811	1.410000006E-08
0.9	0.90492075	0.9049207688	1.880000000E-08
1	1.008333333	1.008333359	2.600000015E-08

CONCLUSION

The 6-step block method proposed for special fourth order initial value problems eliminates the use of predictor-corrector methods and it is also more accurate and faster than the conventional (step-step) integration procedures. In this paper we developed a uniform order 1-block 6- point integrators of orders (4, 4, 4, 4, 4, 4) and the resultant numerical integrators possess the following desirable properties:

- (I) Zero- stability (i.e. stability at the origin).
- (II) Cheap and reliable error estimates.
- (III) Facility to generate the solution at six point simultaneously.
- (IV) It is convergent scheme.

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