The Influence of Denominator Functions on Finite-Difference Schemes in Achieving Correct Linear Stability Properties for a Finite Step-Size.

Joshua Sunday, M.Sc.

Department of Mathematical Sciences,
Adamawa State University, Mubi, Nigeria.

E-mail: joshuasunday2000@yahoo.com

ABSTRACT

This paper presents the role played by denominator functions in achieving correct linear stability of finite-difference schemes of ordinary differential equations. Of particular interest is the work of Mickens (1994). Our goal is to construct finite-difference schemes (with denominator functions) that do not exhibit elementary numerical instabilities for all finite step-sizes.

(Keywords: finite-difference, stability, denominator function, autonomous)

INTRODUCTION

It is a known and documented fact that a differential equation is said to have numerical instabilities if there exist solutions to the finite-difference equations that do not correspond qualitatively to any of the possible solutions of the differential equation (Mickens, 1994). Numerical instabilities are indications that the discrete equations are not able to model the correct mathematical properties of the solutions to the differential equations of interest.

In this work, the equation to be investigated is the autonomous first-order differential equation:

$$\frac{dy}{dx} = f(y) \qquad \textbf{(1)}$$

We shall assume that,

$$f(y) = 0 \qquad \textbf{(2)}$$

has only simple zeros. Our goal is to construct discrete models of Equation (1) that do not exhibit elementary numerical instabilities (i.e., solutions to the finite difference equation that do not correspond to any of the solutions to the differential equation).

It is important to note that numerical instabilities occur by several mechanisms: First, for a central difference scheme, the numerical instabilities are a consequence of the order of the difference scheme being higher than the order of the differential equation. This is illustrated by the use of the decay and logistic equations.

This type of instability occurs because the higher-order difference equation has a larger set of general solution than the corresponding differential equation. For instance, the linear decay equation has only one solution. On the contrary, a discrete model that uses the central difference scheme has two linearly independent solutions since it is of second-order.

Second, most numerical instabilities arise when the step-size is larger than some fixed, finite value, $h > h^* > 0$. All forward Euler type schemes and their generalizations, such as Runge-Kutta methods, have this property.

Third, the implicit backward Euler scheme exhibits super-stability (i.e., its numerical instabilities occur above some threshold value of the step-size, say $h > h_0$, such that all the fixed-point of the difference scheme become stable).

Fourth, the use of higher order (h) schemes, such as Runge-Kutta methods, gives rise to numerical instabilities because of the appearance of spurious real fixed-point for $h > h_1$.

This paper will discuss the denominator function and the new finite difference scheme; the applications of the new finite difference schemes to some chosen problems: and the non-standard modeling rules.

THE DENOMINATOR FUNCTION AND THE NEW FINITE-DIFFERENCE SCHEME

We shall apply the denominator function,

$$D(h, R^*) \equiv \frac{\emptyset(hR^*)}{R^*} \quad \textbf{(3)}$$

to the new finite difference scheme to be developed, where $\emptyset$ *and* R^* are given by,

$$R^* \equiv Max\ \{|R_i|; i = 1,2,\dots,I\} \quad \textbf{(4)}$$

and,

$$\emptyset(Z) = Z + 0(Z^2), Z \to 0 \quad \textbf{(5)}$$

$$0 < \emptyset(Z) < 1, Z > 0 \quad \textbf{(6)}$$

Note that $\emptyset$ has two properties (5) and (6) above. This form replaces the simple "h" function found in the standard finite difference schemes, for example:

$$\frac{dy}{dx} \to \frac{y_{n+1} - y_n}{h}, \quad \textbf{(7)}$$

Note that in the limits ($h \to 0, n \to \infty, hn = x = fixed$), the generalized discrete derivative reduces to the first derivative, for example:

$$\lim_{\substack{h\to 0\\ n\to\infty\\ hn=x=fixed}} \frac{y_{n+1} - y_n}{\left[\frac{\emptyset(hR^*)}{R^*}\right]} = \frac{dy}{dx} \quad \textbf{(8)}$$

However, for fixed $h > 0$ and a given value of n, the generalized discrete derivative may have a numerical magnitude that differs greatly from the standard discrete derivatives such as those given by the central difference, forward Euler and backward Euler representations.

We can consider the denominator function as a **renormalization of the step-size** h to a new valueh', for example:

$$h \to h' = D(hR^*) \quad \textbf{(9)}$$

This concept of renormalized variable and constants occurs frequently in various areas of the sciences (c.f. Caianiello, 1973). It is noteworthy that $\emptyset(Z)$ can be arbitrary. A particularly useful and simple functional form of $\emptyset(Z)$ is the following expression which occurs in many exact finite difference schemes:

$$\emptyset(Z) = 1 - e^{-Z} \quad \textbf{(10)}$$

Let us consider the new finite difference scheme, assume that Equation (1) has fixed-point denoted by:

$$\bar{y}^{(i)}; i = 1,2,3,\dots,I \quad \textbf{(11)}$$

Where I may be unbounded. The fixed-points are the solutions to the equation:

$$f(\bar{y}) = 0 \quad \textbf{(12)}$$

Defined R_i by,

$$R_i \equiv \frac{df[\bar{y}^{(i)}]}{dy}, \quad \textbf{(13)}$$

and R^* is the same as in Equation (4). Linear stability analysis applied to the ith fixed-point gives the following results;

a) If $R_i > 0$, the fixed-point $y(x) = \bar{y}^{(i)}$ is linearly unstable

b) If $R_i < 0$, the fixed-point $y(x) = \bar{y}^{(i)}$ is linearly stable

Hence, the finite difference scheme for Equation (1) is:

$$\frac{y_{n+1} - y_n}{\left[\frac{\emptyset(hR^*)}{R^*}\right]} = f(y_n) \quad \textbf{(14)}$$

Where $\emptyset(Z)$ have two properties as in (5) and (6).

<u>Theory 1 (Mickens, 1994)</u>

The finite difference scheme (14) has fixed points with exactly the same linear stability properties as the differential equation (1) for all$h > 0$.

<u>Proof</u>

Let a perturbation about the ith fixed point be denoted by,

$$y_n = \bar{y}^{(i)} + \epsilon_n \quad \textbf{(15)}$$

The linear stability analysis equation for ϵ_n is:

$$\frac{\epsilon_{n+1} - \epsilon_n}{\left[\frac{\emptyset(hR^*)}{R^*}\right]} = R_i\ \epsilon_n, \quad \textbf{(16)}$$

or,

$$\epsilon_{n+1} = [1 + (\frac{R_i}{R^*})\emptyset(hR^*)]\ \epsilon_n, \quad \textbf{(17)}$$

which has the solution,

$$\epsilon_n = \epsilon_0\ [1 + (\frac{R_i}{R^*})\emptyset(hR^*)]^n \quad \textbf{(18)}$$

If $R_i > 0$, the fixed point of the differential equation is linearly unstable. Thus, it follows that,

$$1 + \left(\frac{R_i}{R^*}\right)\emptyset(hR^*) > 0, h > 0 \quad \textbf{(19)}$$

Therefore, $y_n = \bar{y}^{(i)}$ is also linearly unstable for $h > 0$. If $R_i > 0$, the fixed point of the differential equation is linearly stable. In this instance,

$$0 < 1 - \left(\frac{R_i}{R^*}\right)\emptyset(hR^*) < 1\ , h > 0 \quad \textbf{(20)}$$

a result that follows directly from (4), (5), and (6).

Therefore $y_n = \bar{y}^{(i)}$ linearly stable for $h > 0$. This theorem shows that it is possible to construct discrete models for a single scalar, autonomous ordinary differential equation such that no elementary numerical instabilities occur in their solutions. This result is related to the fact that most elementary numerical instabilities arise from a given fixed point having the opposite linear stability properties in the difference scheme and the differential equation.

The above construction demonstrates that to achieve the correct linear stability behavior, a generalized definition of the discrete derivative must be used (Mickens, 1994). None of the standard finite difference modeling procedures has this property, namely, the correct linear stability behavior for all step-sizes. Thus, the finite difference scheme of Equation (14) has fixed point with exactly the same linear stability properties as the scalar differential equation. This result holds for all finite step sizes, $h > 0$. consequently, "super-stability'' will not occur in the discrete model of Equation (14).

APPLICATIONS OF THE NEW FINITE DIFFERENCE SCHEME

The power of the new finite difference scheme in Equation (14) shall be illustrated by applying it to three equations namely; the decay equation, the logistic equation and an equation having three fixed-point.

Example 1 (Decay Equation)

The decay differential equation is,

$$\frac{dy}{dx} = -\lambda y \quad \textbf{(21)}$$

where λ is a positive constant. Comparing (21) with (1), we see that,

$$f(y) = -\lambda y \quad \textbf{(22)}$$

A single globally stable fixed point exists at $\bar{y}^{(i)}$=0. In addition,

$$R_1 = -\lambda, R^* = \lambda \quad \textbf{(23)}$$

Now, select for $\emptyset(Z)$, with the expression,

$$\emptyset(Z) = 1 - e^{-z} \quad \textbf{(24)}$$

We now substitute these results in Equation (14) to get,

$$\frac{y_{n+1} - y_n}{\left(\frac{1-e^{-\lambda h}}{\lambda}\right)} = -\lambda y_n \quad \textbf{(25)}$$

This new finite difference scheme gives the exact discrete model.

Example 2 (Logistic Equation)

Consider the logistic differential equation,

$$\frac{dy}{dx} = y(1-y) \quad \textbf{(26)}$$

Comparing equation (26) with Equation (1), we see that,

$$f(y) = y(1-y) \quad \textbf{(27)}$$

Two fixed points exists at,

$$\bar{y}^{(1)} = 0, \bar{y}^{(2)} = 1 \quad \textbf{(28)}$$

and

$$R_1 = 1, R_2 = -1, R^* = 1 \quad \textbf{(29)}$$

Substituting Equations (27), (29), and (24) into Equation (14) gives,

$$\frac{y_{n+1}-y_n}{1-e^{-h}} = -y_n(1-y_n) \quad \textbf{(30)}$$

Example 3 (ODE with Three Fixed-Points)

The simplest ordinary differential equation with three fixed-points is,

$$\frac{dy}{dx} = y(1-y^2) \quad \textbf{(31)}$$

It is evident that, from Equation (31),

$$f(y) = y(1-y^2) \quad \textbf{(32)}$$

$$\bar{y}^{(1)} = 0, \bar{y}^{(2)} = 1, \bar{y}^{(3)} = -1 \quad \textbf{(33)}$$

$$R_1 = 1, R_2 = R_3 = -2, R^* = 2 \quad \textbf{(34)}$$

Using $\emptyset(Z)$ from Equation (24), we obtain, on substitution of these results into Equation (14), the following discrete model for Equation (31),

$$\frac{y_{n+1}-y_n}{\left(\frac{1-e^{-2h}}{2}\right)} = y_n(1-y_n^2) \quad \textbf{(35)}$$

We have seen from the three finite-difference schemes constructed above for decay Equation (25), logistic equation (30), and ODE with three fixed-points (35), that using the denominator function in (3), it is possible to construct finite difference schemes with correct linear stability.

THE NON-STANDARD MODELLING RULES

We present below the non-standard modeling rules based on Mickens (1994).

Rule 1

The order of the discrete derivatives must be exactly equal to the order of the corresponding derivatives of the differential equations. If this rule is violated, this can lead to numerical instability in the form of oscillations which may be bounded or unbounded. The mathematical reason for the above occurrence is that discrete equations have large class of solutions than differential equations.

As an illustration, let us consider the following first order differential equation,

$$\frac{dy}{dx} = -y \quad \textbf{(36)}$$

If we model (36) by central difference scheme of the form,

$$\frac{y_{n+1}-y_n}{2h} = -y_n \quad \textbf{(37)}$$

It will be discovered that this modeling has extra solution that is strange because Equation (37) is of second order while (36) is of first order and this leads to instability.

Rule 2

Denominator functions for the discrete derivatives must be expressed in terms of more complicated functions of the step-sizes than those conventionally used. This rule allows the introduction of complex analytic function of h in the denominator. For example, let us consider,

$$\frac{dy}{dx} = y(1-y) \quad \textbf{(38)}$$

This is in form of a logistic equation. If we choose, $D_1 = e^h - 1, h = D_x$ we have,

$$\frac{y_{n+1}-y_n}{e^h-1} = y_n(1-y_n) \quad \textbf{(39)}$$

It must be stated here that selection of an appropriate denominator is an 'art' (Mickens, 1990).

We must examine the differential equations for which the exact schemes are known. Close examination of the differential equations for which exact schemes are known, shows, that the denominator functions generally are functions that are related to particular solution or properties of the general solution to the difference equation.

Rule 3

Non-linear terms must, in general, be modeled non-locally on the computational grid or lattice.

Rule 4

Special solutions of the differential equations should also be special (discrete) solutions of the finite difference models.

Rule 5

The finite difference equations should not have solutions that do not correspond exactly to solutions of the differential equations.

We now take a close look at Rule 3, which has the requirement that non linear terms be modeled non-locally on the computational grid. Applications will be made on two differential equations.

Logistic Equation

The discrete scheme for the Logistic differential equation, with a non linear term is,

$$\frac{y_{n+1}-y_n}{1-e^{-h}} = y_n(1 - y_{n+1}) \quad \textbf{(40)}$$

Using transformation, this difference equation can be solved exactly to get,

$$y_n = \frac{1}{\omega_n} \quad \textbf{(41)}$$

This gives,

$$\omega_{n+1} - \left(\frac{1}{2-e^{-h}}\right)\omega_n = \frac{1-e^{-h}}{2-e^{-h}} \quad \textbf{(42)}$$

with exact solution,

$$\omega_n = 1 + A(2 - e^{-h})^{-n} \quad \textbf{(43)}$$

where, A is an arbitrary constant. Imposing the initial condition, $y(0) = y_0$, we obtain:

$$y_n = \frac{y_0}{y_0+(1-y_0)(2-e^{-h})^{-h}} \quad \textbf{(44)}$$

Note that,

$$1 < 2 - e^{-h} < 2, h > 0, \quad \textbf{(45)}$$

Consequently,

$$g_n = (2 - e^{-h})^{-n} \quad \textbf{(46)}$$

is an exponentially decreasing function of n. Examination of Equation (14) shows that all the solutions of Equation (40) have the same qualitative properties as the solutions to the Logistic differential equation for all step sizes, $h > 0$.

ODE with Three Fixed-Points

A discrete scheme for equation (31), with a non-local non-linear term, is

$$\frac{y_{n+1}-y_n}{\left(\frac{1-e^{-2h}}{2}\right)} = y_n(1 - y_{n+1}y_n) \quad \textbf{(47)}$$

This expression is linear in y_{n+1}, solving for it gives

$$y_{n+1} = \frac{(1+\emptyset)y_n}{1+\emptyset y_n^2} \quad \textbf{(48)}$$

where,

$$\emptyset = \frac{1-e^{-2h}}{2} \quad \textbf{(49)}$$

From all our presentations above, we have been able to construct models using denominator functions that have exactly the same qualitative behavior as their corresponding solutions to the differential equations.

CONCLUSION

We have presented finite difference schemes with correct linear stability properties using denominator functions. In other words this paper has emphasized that the use of renormalized function has a more important effect in achieving a correct linear stability property on finite difference schemes.

REFERENCES

1. Humi, M. and Miller, W. 1988. *Second Course in Ordinary Differential Equations for Scientists and Engineers*. Springer- Verlag: New York, NY.

2. Mickens, R.E. 1990. *Difference Equation; Theory and Applications*. Van Non-Strand Reinhold: New York, NY.

3. Mickens, R.E. 1994. “Non-Standard Finite Difference Models of Differential Equations”. *World Scientific*. Singapore.

4. Rose, S.L. 1964. *Differential Equations*. Blaisdeu: Waltham, MA.

ABOUT THE AUTHOR

Joshua Sunday is a Lecturer at the Adamawa State University, Mubi-Nigeria. He holds a Masters of Science (M.Sc.) degree in Numerical Analysis. His research interests are in numerical analysis.

SUGGESTED CITATION

Sunday, J. 2010. “The Influence of Denominator Functions on Finite-Difference Schemes in Achieving Correct Linear Stability Properties for a Finite Step-Size”. *Pacific Journal of Science and Technology*. 11(1):239-244.

Pacific Journal of Science and Technology