On Equivalence of n-Norms in n-Normed Spaces.

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ABSTRACT

In this paper, we investigate some properties of linear n-normed spaces and obtained necessary and sufficient conditions for n-norms to be equivalent on linear n-normed spaces.

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INTRODUCTION

In [6, 7] Gähler introduced an attractive theory of 2-norm and n-norm on a linear space. Raymond W. Freese and Y.J. Cho [4] introduced as a survey of the latest results on the relations between linear 2-normed spaces and normed linear spaces, completion of linear 2-normed spaces. A systematic development of linear n-normed spaces has been extensively made by S.S. Kim and Y.J. Cho [8], R. Malceski [5], A. Misiak [1] and Hendra Gunawan and Mashadi [3]. In this paper, some necessary and sufficient conditions for n-norms to be equivalent on a linear n-normed space are given.

Let $n \in N$ and let X be a real linear space of dimension $d \geq n$. A real valued function $\|\bullet,\bullet,\ldots,\bullet\| : X^n \to R$ satisfying the following four properties:

nN_1: $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent vectors

nN_2: $\|x_1, x_2, \ldots, x_n\| = \|x_{j_1}, x_{j_2}, \ldots, x_{j_n}\|$ for every permutation $(j_1, j_2, \ldots, j_n)$ of $(1,2,\ldots,n)$ *i.e.*, $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.

nN_3 : $\|x_1, x_2, \ldots, x_{n-1}, \alpha x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in R$

nN_4 : $\|x_1, x_2, \ldots, x_{n-1}, y+z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$ for all $y, z, x_2, \ldots, x_{n-1} \in X$, is called an n-norm on X and the pair $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ is called a linear n-normed space.

Example 1: [3] Let $X = R^n$. Let us define the function ||.,…,.|| on X by
$\|x_1, x_2, \ldots, x_n\| = |\det(x_{ij})|$,

$$= abs\left(\begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix}\right)$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ for each $i = 1,2,\ldots,n$. Then $(X, \|\bullet,\bullet,\ldots,\bullet\|)$ is a linear n-normed space.

Example 1: Consider the linear space P_m of real polynomials of degree $\leq m$ on the interval $[0,1]$. Let $x_i {}_{i=0}^{nm}$ be $nm+1$ arbitrary but distinct fixed points in $[0,1]$. For $f_1, f_2, \ldots, f_n$ in P_m, let us define
$\|f_1, f_2, \ldots, f_n\| = 0$, if $f_1, f_2, \ldots, f_n$ are linearly dependent,

$$\|f_1, f_2, \dots, f_n\| = \sum_{i=0}^{nm} |f_1(x_i) f_2(x_i) \dots f_n(x_i)|,$$

if $f_1, f_2, \dots, f_n$ are linearly independent.

Then $\|\bullet, \bullet, \dots, \bullet\|$ is an *n*-norm on P_m.

Solution: If $f_1, f_2, \dots, f_n$ are linearly dependent, then $\|f_1, f_2, \dots, f_n\| = 0$. Conversely assume $\sum_{i=0}^{nm} |f_1(x_i) f_2(x_i) \dots f_n(x_i)| = 0$ This implies that $f_1(x_i) f_2(x_i) \dots f_n(x_i) = 0$ at $nm+1$ distinct points. Since the degree of each $f_i \le m$, we must have at least one $f_i = 0$.Thus $\|f_1, f_2, \dots, f_n\| = 0$ if and only if $f_1, f_2, \dots, f_n$ are linearly dependent. Other properties of *n*-norm can be verified easily.

A sequence $\{x_k\}$ in a linear *n*-normed space $(X, \|\bullet, \bullet, \dots, \bullet\|)$ is called convergent to x if $\lim_{k \to \infty} \|x_k - x, w_2, w_3, \dots, w_n\| = 0$ for all $w_2, w_3, \dots, w_n \in X$. It is denoted by $x_k \to x$ as $k \to \infty$

A sequence $\{x_k\}$ in a linear *n*-normed space $(X, \|\bullet, \bullet, \dots, \bullet\|)$ is called Cauchy sequence if $\lim_{k,m \to \infty} \|x_k - x_m, w_2, w_3, \dots, w_n\| = 0$ for all $w_2, w_3, \dots, w_n \in X$.

A linear *n*-normed space X is said to be complete if every Cauchy sequence in X is convergent. A complete *n*-normed space is said to be an *n*-Banach space.

SOME ELEMENTARY PROPERTIES

Proposition 1: Let $(X, \|\bullet, \bullet, \dots, \bullet\|)$ be a linear *n*-normed space.

(*i*) If $\{x_k\}$ is a Cauchy sequence in X, then $\|x_k, w_2, w_3, \dots, w_n\| : w_2, w_3, \dots, w_n \in X, k \in N$ is a Cauchy sequence of non-negative reals.

(*ii*) If $\{x_k\}$ and $\{y_k\}$ are Cauchy sequences in X and $\{\alpha_k\}$ is a Cauchy sequence of reals then $\{x_k + y_k\}$ and $\{\alpha_k x_k\}$ are Cauchy sequences in X.

Proof: (*i*) Let $\{x_k\}$ is a Cauchy sequence in X. Then $\lim_{k,m \to \infty} \|x_k - x_m, w_2, w_3, \dots, w_n\| = 0$, for all $w_2, w_3, \dots, w_n \in X$. We have

$\|x_k, w_2, w_3, \dots, w_n\|$
$= \|(x_k - x_m) + x_m, w_2, w_3, \dots, w_n\|$
$\le \|x_k - x_m, w_2, w_3, \dots, w_n\| + \|x_m, w_2, w_3, \dots, w_n\|$
(by nN_4) $\|x_k, w_2, w_3, \dots, w_n\|$ - $\|x_m, w_2, w_3, \dots, w_n\| \le \|x_k - x_m, w_2, w_3, \dots, w_n\|$

Similarly, we have
$\|x_m, w_2, w_3, \dots, w_n\|$ - $\|x_k, w_2, w_3, \dots, w_n\|$
$\le \|x_k - x_m, w_2, w_3, \dots, w_n\|$

Combining the above inequalities, we have
$| \, \|x_k, w_2, w_3, \dots, w_n\| - \|x_m, w_2, w_3, \dots, w_n\| \, |$
$\le \|x_k - x_m, w_2, w_3, \dots, w_n\| \to 0$ as $k, m \to \infty$.

Therefore,
$| \, \|x_k, w_2, w_3, \dots, w_n\| - \|x_m, w_2, w_3, \dots, w_n\| \, | \to 0$
as $k, m \to \infty$.

Hence, $\|x_k, w_2, w_3, \dots, w_n\|$ w_2 w_3 $, \dots, w_n \in$, is a Cauchy sequence of non-negative reals.
(*ii*) Let $\{x_k\}$ and $\{y_k\}$ be two Cauchy sequences in X.

Then,
$\lim_{k,m\to\infty}\|x_k - x_m, w_2, w_3,..., w_n\| = 0$, for all $w_2, w_3,..., w_n \in X$ and
$\lim_{k,m\to\infty}\|y_k - y_m, w_2, w_3,..., w_n\| = 0$, for all $w_2, w_3,..., w_n \in X$.

Now,

$$\|(x_k + y_k) - (x_m + y_m), w_2, w_3,..., w_n\|$$
$$= \|(x_k - x_m) + (y_k - y_m), w_2, w_3,..., w_n\|$$
$$\leq \|x_k - x_m, w_2, w_3,..., w_n\| + \|y_k - y_m, w_2, w_3,..., w_n\|$$
$$\to 0 \text{ as } k, m \to \infty.$$

Hence, $\{x_k + y_k\}$ is a Cauchy sequence in X.

Let $\{\alpha_k\}$ be a Cauchy sequence of reals. Also from (*i*), we have $\|x_k, w_2, w_3,..., w_n\| : w_2, w_3,..., w_n \in X$ is a Cauchy sequences of reals. Hence they are bounded. We can find K_1, $K_2 \geq 0$ such that $|\alpha_k| \leq K_1$ and $\|x_k, w_2, w_3,..., w_n\| \leq K_2$ for all $k \in N$.

We have $\|\alpha_k x_k - \alpha_m x_m, w_2, w_3,..., w_n\|$

$$= \|\alpha_k x_k - \alpha_k x_m + \alpha_k x_m - \alpha_m x_m, w_2, w_3,..., w_n\|$$
$$\leq \|\alpha_k x_k - \alpha_k x_m, w_2, w_3,..., w_n\| + \|\alpha_k x_m - \alpha_m x_m, w_2, w_3,..., w_n\|$$
$$= |\alpha_k| \; \|x_k - x_m, w_2, w_3,..., w_n\| + |\alpha_k - \alpha_m| \; \|x_m, w_2, w_3,..., w_n\|$$
$$\leq K_1 \; \|x_k - x_m, w_2, w_3,..., w_n\| + K_2 \; |\alpha_k - \alpha_m|$$
$$\to 0, \text{ as } k, m \to \infty.$$

Thus, $\|\alpha_k x_k - \alpha_m x_m, w_2, w_3,..., w_n\| \to 0$ as $k, m \to \infty$ Hence $\{\alpha_k x_k\}$ is a Cauchy sequence in X.

Proposition 2: In any linear *n*-normed space $(X, \|\bullet,\bullet,...,\bullet\|)$, we have the following

(*i*) If $x_k \to x$ and $y_k \to y$ as $k \to \infty$ then $x_k + y_k \to x + y$ as $k \to \infty$

(*ii*) If $x_k \to x$ and $\alpha_k \to \alpha$ as $k \to \infty$ then $\alpha_k x_k \to \alpha x$ as $k \to \infty$

(*iii*) If $\dim X \geq n$ and $x_k \to x$, $y_k \to y$ as $k \to \infty$ then $x = y$

Proof : (*i*) we have

$$\|(x_k + y_k) - (x + y), w_2, w_3,..., w_n\|$$
$$= \|(x_k - x) + (y_k - y), w_2, w_3,..., w_n\|$$
$$\leq \|x_k - x, w_2, w_3,..., w_n\| + \|y_k - y, w_2, w_3,..., w_n\|$$
$$\to 0 \text{ as } k \to \infty$$

Therefore, $x_k + y_k \to x + y$ as $k \to \infty$

(*ii*) Using the fact that a real convergent sequence is bounded, we have:

$$\|\alpha_k x_k - \alpha x, w_2, w_3,..., w_n\|$$
$$= \|\alpha_k x_k - \alpha_k x + \alpha_k x - \alpha x, w_2, w_3,..., w_n\|$$
$$\leq \|\alpha_k x_k - \alpha_k x, w_2, w_3,..., w_n\| + \|\alpha_k x - \alpha x, w_2, w_3,..., w_n\|$$
$$= |\alpha_k| \; \|x_k - x, w_2, w_3,..., w_n\| + |\alpha_k - \alpha| \; \|x, w_2, w_3,..., w_n\|$$
$$\leq K \; \|x_k - x, w_2, w_3,..., w_n\| + |\alpha_k - \alpha| \; \|x, w_2, w_3,..., w_n\|$$ for some $K \geq 0$.

Therefore, $\alpha_k x_k \to \alpha x$ as $k \to \infty$, since $\lim_{k\to\infty}\|x_k - x, w_2, w_3,..., w_n\| = 0$, $\lim_{k\to\infty}|\alpha_k - \alpha| = 0$ and $\|x, w_2, w_3,..., w_n\|$ is finite.

(*iii*) We can write for each $k \in N$ and $w_2, w_3,..., w_n \in X$, $\|x - y, w_2, w_3,..., w_n\|$

$$= \|x - x_k + x_k - y, w_2, w_3,..., w_n\|$$
$$\leq \|x - x_k, w_2, w_3,..., w_n\| + \|x_k - y, w_2, w_3,..., w_n\|$$

It follows that:
$\|x - y, w_2, w_3,..., w_n\| = 0$ for all $w_2, w_3,..., w_n \in X$, since $x_k \to x$, $y_k \to y$ as $k \to \infty$. Hence $x - y, w_2, w_3,..., w_n$ are linearly dependent for all $w_2, w_3,..., w_n \in X$. Since

$\dim X \geq n$, the only way that $x-y$ can be linearly dependent with all vectors $w_2, w_3, ..., w_n \in X$ is for $x-y=0 \Rightarrow x=y$.

Proposition 3: Let $(X, \|\bullet,\bullet,...,\bullet\|)$ be a linear *n*-normed space. If x_k be a Cauchy sequence in *X*, then, $\|x_k - x, w_2, w_3, ..., w_n\| : w_2, w_3, ..., w_n \in X, k \in N$ is a Cauchy sequence of non-negative reals for each $x \in X$.

Proof: We have $\|x_k - x, w_2, w_3, ..., w_n\|$ =

$$\|x_k - x_m + x_m - x, w_2, w_3, ..., w_n\|$$

$$\leq \|x_k - x_m, w_2, w_3, ..., w_n\| + \|x_m - x, w_2, w_3, ..., w_n\|$$

$$\|x_k - x, w_2, w_3, ..., w_n\| - \|x_m - x, w_2, w_3, ..., w_n\|$$

$$\leq \|x_k - x_m, w_2, w_3, ..., w_n\|.$$

Similarly, we have,

$$\|x_m - x, w_2, w_3, ..., w_n\| - \|x_k - x, w_2, w_3, ..., w_n\|$$

$$\leq \|x_k - x_m, w_2, w_3, ..., w_n\|$$

Combining the above inequalities, we have,

$$\left| \|x_k - x, w_2, w_3, ..., w_n\| - \|x_m - x, w_2, w_3, ..., w_n\| \right|$$

$$\leq \|x_k - x_m, w_2, w_3, ..., w_n\|$$

$$\left| \|x_k - x, w_2, w_3, ..., w_n\| - \|x_m - x, w_2, w_3, ..., w_n\| \right|$$

as $k, m \to \infty$, since $\{x_k\}$ is a Cauchy sequence.

Hence, $\|x_k - x, w_2, w_3, ..., w_n\| : w_2, w_3, ..., w_n \in X$ is Cauchy sequence of non-negative reals for each $x \in X$.

Proposition 4: If $\lim_{k\to\infty} \|x_k - x, w_2, w_3, ..., w_n\|$ = 0 then $\lim_{k\to\infty} \|x_k, w_2, w_3, ..., w_n\|$ = $\|x, w_2, w_3, ..., w_n\|$.

Proof: Let $\lim_{k\to\infty} \|x_k - x, w_2, w_3, ..., w_n\| = 0$.

We have,

$$\left| \|x_k, w_2, w_3, ..., w_n\| - \|x, w_2, w_3, ..., w_n\| \right|$$

$$\leq \|x_k - x, w_2, w_3, ..., w_n\|$$

It follows that,

$$\left| \|x_k, w_2, w_3, ..., w_n\| - \|x, w_2, w_3, ..., w_n\| \right| \to 0$$

as $k \to \infty$

Hence,

$$\lim_{k\to\infty} \|x_k, w_2, w_3, ..., w_n\| = \|x, w_2, w_3, ..., w_n\|$$

Proof of the following two Propositions is easy, so omitted. For some similar results on *n*-inner product spaces, one may refer to Hendra Gunawan [2].

Proposition 5: *Limit of every convergent sequence in an n-normed space is unique*.

Proposition 6: *Every convergent sequence in an n-normed space is a Cauchy sequence*.

Now we are ready to give the main Theorem of this paper.

MAIN RESULTS

In this section we prove necessary and sufficient conditions for *n*-norms to be equivalent on linear *n*-normed spaces.

Definition 1: Two *n*-norms $\|\bullet,\bullet,...,\bullet\|_1$ and $\|\bullet,\bullet,...,\bullet\|_2$ on a linear *n*-normed space X are said to be equivalent if there exists constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha \|a, w_2, ..., w_n\|_1 \leq \|a, w_2, ..., w_n\|_2 \leq \beta \|a, w_2, ..., w_n\|_1$$

$$\forall\ a, w_2, ..., w_n \in X.$$

Theorem 1: Two *n*-norms $\|\bullet,\bullet,...,\bullet\|_1$ and $\|\bullet,\bullet,...,\bullet\|_2$ are equivalent on a linear *n*-normed space if and only if every Cauchy sequence with respect to one of the *n*-norms is a Cauchy sequence with respect to other *n*-norm.

Proof: Suppose that two n-norms $\|\bullet,\bullet,...,\bullet\|_1$ and $\|\bullet,\bullet,...,\bullet\|_2$ are equivalent on a linear n-normed space X. Then there exists constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\|a,w_2,...,w_n\|_1 \le \|a,w_2,...,w_n\|_2 \le \beta\|a,w_2,...,w_n\|_1$$

$\forall\ a,w_2,...,w_n \in X$. For a sequence $\{x_k\}$ in X we have

(1) $$\alpha\|x_k - x_m, w_2,...,w_n\|_1 \le \|x_k - x_m, w_2,...,w_n\|_2 \le \beta\|x_k - x_m, w_2,...,w_n\|_1$$

for all $w_2, w_3,..., w_n \in X$ and $k,m \in N$.

The second inequality shows that if $\{x_k\}$ is Cauchy sequence with respect to $\|\bullet,\bullet,...,\bullet\|_1$ if and only if it is a Cauchy sequence with respect to $\|\bullet,\bullet,...,\bullet\|_2$.

For the converse part, suppose that the n-norms are not equivalent. Then without loss of generality we can assume the following two cases:

(i) we cannot find α such that

$$\alpha\|a,w_2,...,w_n\|_1 \le \|a,w_2,...,w_n\|_2$$

$\forall\ a,w_2,...,w_n \in X$.

(*ii*) we cannot find β such that

$$\|a,w_2,...,w_n\|_2 \le \beta\|a,w_2,...,w_n\|_1$$

$\forall\ a,w_2,...,w_n \in X$.

In case (*i*) for k =1, 2, … , there exists x_k in X such that

(2) $$\frac{1}{k}\|x_k, w_2,...,w_n\|_1 > \|x_k, w_2,...,w_n\|_2$$

Let $y_k = \frac{1}{\sqrt{k}}\frac{1}{\|x_k, w_2,...,w_n\|_2} x_k$, for each $k \in N$.

Then,

$$\|y_k, w_2,...,w_n\|_2 = \frac{1}{\sqrt{k}} \to 0 \text{ as } k \to \infty$$

and using **(2)** we get

$$\|y_k, w_2,...,w_n\|_1 = \frac{1}{\sqrt{k}}\frac{1}{\|x_k, w_2,...,w_n\|_2}\|x_k, w_2,...,w_n\|_1$$

$$> \frac{k}{\sqrt{k}} = \sqrt{k} \to \infty \text{ as } k \to \infty$$

So, using Proposition 6, $\{y_k\}$ is a Cauchy sequence with respect to $\|\bullet,\bullet,...,\bullet\|_2$ but not with respect to $\|\bullet,\bullet,...,\bullet\|_1$. Similarly, we can prove case (*ii*). Hence the theorem.

Corollary 1: Let $\|\bullet,\bullet,...,\bullet\|_1$ and $\|\bullet,\bullet,...,\bullet\|_2$ be two equivalent n-norms on a linear n-normed space X, then $x_k \to x$ with respect to $\|\bullet,\bullet,...,\bullet\|_1$ if and only if $x_k \to x$ with respect to $\|\bullet,\bullet,...,\bullet\|_2$.

Proof: By replacing '$x_k - x_m$' with '$x_k - x$' in **(1)** of Theorem 1, we get the result.

CONCLUSIONS

The concept of 2-normed spaces was introduced and studied by Siegfried Gähler, a German mathematician who worked at the German Academy of Science, Berlin, in a series of paper in German language published in *Mathematische Nachrichten* in the mid of 1960's. Later, it was further generalized and introduced the notion of *n*-norm by Misiak.

Very often Gähler has raised the following questions: What is the real motivation for studying 2-norm structure? Is there a physical situation or an abstract concept where norm topology does not work but 2-norm topology does work?

After going through the results of this paper, we see that while studying *n*-norm structure the main issue should be the use of the meaning of *n*-norms. We also observe that if a term in the definition of *n*-norm represents the change of shape, and the *n*-norm stands for the associated area or center of gravity of the term, maybe we can think of some plausible applicable of the notion of *n*-norm, and then the generalized convergence make sense. This can also be viewed as: Suppose for a particular output we need *n*-inputs but with one main input and other (*n*-1)-inputs are required to complete the process.

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