An Accurate Uniform Order 6 Block Method for Direct Solution of General Second Order Ordinary Differential Equations.

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ABSTRACT

In this Paper, we extend the idea of collocation of linear multistep methods to develop a uniform Order 6 5step block methods. The single continuous formulation derived is evaluated at grid points of,

$$x = x_{n+j}\ , j = 2, 3, (k)$$

and its first derivative was also evaluated at some grid points,

$$x = x_{n+j}\ , j = 0, 1, 2, (k)\ ,$$

yielded the multi discrete schemes that form a Self starting uniform order 6 block methods. Two Numerical examples which are linear and non linear differential equations were used to demonstrate the efficiency of the methods.

(Keywords: linear multi-step methods, LMM, block methods, zero stable, continuous formulation, CF)

INTRODUCTION

In this article, efforts are directed towards constructing a uniform order 6 block methods for solution of general second order ordinary differential equations of the form:

$$y'' = f(x, y, y') \qquad y(0) = \propto\ ,\ y'(0) = \beta \qquad (1)$$

In the past, efforts have been made by eminent Scholars to solve higher order initial value problems especially the Second order ordinary differential equations. In practice, this class of problem (1) are usually reduced to system of first order differential equations and numerical methods for first order Odes is then employ to solve them see Fatunla(1988) and Lambert (1973).

Awoyemi (1999) showed that reduction of higher order equations to its first order has a serious implication in the results , hence it is necessary to modify existing algorithms to handle directly this class of problem (1). In a recent paper of Yahaya and Badmus (2009), authors demonstrated a successful application of LMM methods to solve directly a general second order Odes of form (1) though with non uniform order member Block method and this idea is now extended to these uniform order block schemes to solve the type (1) directly.

DERIVATION OF THE NEW BLOCK SCHEMES

We propose an approximate solution to (1) in the form:

$$y_k(x) = \sum_{j=0}^{m+t-1} a_j x^j \qquad (2)$$

$$y'_k(x) = \sum_{j=0}^{m+t-1} j a_j\ x^j \qquad (3)$$

$$y''_k(x) = \sum_{j=0}^{m+t-1} j(j-1) a_j\ x^{j-2} = f(x, y, y') \qquad (4)$$

and interprolate (2) at

$$x = x_{n+j}\ , j = 0, 1, 2, (k).$$

if we collocate (4). Where t and m are points of interpolation and

$$x = x_{n+j}\ ,\ j = 0, 1.$$

and yields the k=5 collocation and specifically for this method following system on non-linear equations.

$$\begin{aligned}
&a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 + a_6 x_n^6 + a_7 x_n^7 = y_n \\
&a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + a_4 x_{n+1}^4 + a_5 x_{n+1}^5 + a_6 x_{n+1}^6 + \\
&\qquad a_7 x_{n+1}^7 = y_{n+3} \\
&2a_2 + 6a_3 x_n + 12a_4 x_n^2 + 20a_5 x_n^3 + 30a_6 x_n^4 + 42x_n^5 = f_n \\
&2a_2 + 6a_3 x_{n+1} + 12a_4 x_{n+1}^2 + 20a_5 x_{n+1}^3 + 30a_6 x_{n+1}^4 + 42a_7 x_{n+1}^5 = f_{n+1} \\
&2a_2 + 6a_3 x_{n+2} + 12a_4 x_{n+2}^2 + 20a_5 x_{n+2}^3 + 30a_6 x_{n+2}^4 + 42x_{n+2}^5 = f_{n+2} \\
&2a_2 + 6a_3 x_{n+3} + 12a_4 x_{n+3}^2 + 20a_5 x_{n+3}^3 + 30a_6 x_{n+3}^4 + 42x_{n+3}^5 = f_{n+3} \\
&2a_2 + 6a_3 x_{n+4} + 12a_4 x_{n+4}^2 + 20a_5 x_{n+4}^3 + 30a_6 x_{n+4}^4 + 42x_{n+4}^5 = f_{n+4} \\
&2a_2 + 6a_3 x_{n+5} + 12a_4 x_{n+5}^2 + 20a_5 x_{n+5}^3 + 30a_6 x_{n+5}^4 + 42x_{n+5}^5 = f_{n+5}
\end{aligned} \tag{5}$$

Where a_j are the parameters to be determined. When rearranging Equation (5) in a matrix equation form, we have:

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \\
0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 \\
0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 & 30x_{n+3}^4 & 42x_{n+3}^5 \\
0 & 0 & 2 & 6x_{n+4} & 12x_{n+4}^2 & 20x_{n+4}^3 & 30x_{n+4}^4 & 42x_{n+4}^5 \\
0 & 0 & 2 & 6x_{n+5} & 12x_{n+5}^2 & 20x_{n+5}^3 & 30x_{n+5}^4 & 42x_{n+5}^5
\end{bmatrix}
\begin{vmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{vmatrix}
=
\begin{vmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{vmatrix} \tag{6}$$

are obtained as continuous coefficients (x) ***and*** $\beta_j\ (x)^{\propto_j}$. Specifically, the proposed solution takes the form:

$$\begin{aligned}
&y(x) = \propto_0 (x) y_n + \propto_1 (x) y_{n+1} + h^2 \{\beta_0(x) f_n + \beta_1(x) f_{n+1} + \\
&\beta_2(x) f_{n+2} + \beta_3(x) f_{n+3} + \beta_4(x) f_{n+4} + \beta_5(x) f_{n+5}\}
\end{aligned} \tag{7}$$

Using Maple 11 Mathematical Soft ware to invert the matrix D in Equation (6) and obtain values for a_j $, j = 0, 1, k+2$ and we obtain the continuous formulation of the form.

$$
\begin{aligned}
y(x) = {} & \left[\frac{h-(x-x_n)}{h}\right] y_n + \left[\frac{(x-x_n)}{h}\right] y_{n+1} \quad + [-2462h^6(x-x_n) + \\
& 5040h^5(x-x_n)^2 - 3836h^4(x-x_n)^3 + 1575h^3(x-x_n)^4 - \\
& 357h^2(x-x_n)^5 + 42h(x-x_n)^6 - 2(x-x_n)^7] \frac{f_n}{10080h^5} + \\
& [-4315h^6(x-x_n) + 8400h^4 (x-x_n)^3 \; - 5390h^3 (x-x_n)^4 \\
& 1491h^2 (x-x_n)^5 - 196h(x-x_n)^6 + 10(x-x_n)^7] \frac{f_{n+1}}{10080h^5} + \\
& [1522h^6(x-x_n) - 4200h^4(x-x_n)^3 + 3745h^3(x-x_n)^4 - \\
& 1239h^2(x-x_n)^5 + 182h(x-x_n)^6 - 10(x-x_n)^7] \frac{f_{n+2}}{5040h^5} + \\
& [-941h^6(x-x_n) + 2800h^4 (x-x_n)^3 \; - 2730h^3(x-x_n)^4 + \\
& 1029h^2(x-x_n)^5 - 168h(x-x_n)^6 + 10(x-x_n)^7] \frac{f_{n+3}}{5040h^5} + \\
& [682h^6(x-x_n) - 2100h^4 (x-x_n)^3 \; + 2135h^3(x-x_n)^4 - \\
& 861h^2(x-x_n)^5 + 154h(x-x_n)^6 - 10(x-x_n)^7] \frac{f_{n+4}}{10080h^5} + \\
& [-107h^6(x-x_n) + 336h^4 (x-x_n)^3 \; - 350h^3(x-x_n)^4 + \\
& 147h^2(x-x_n)^5 - 28h(x-x_n)^6 + 2(x-x_n)^7] \frac{f_{n+5}}{10080h^5}
\end{aligned}
\tag{8}
$$

Evaluating the equation (8) at $x = x_{n+j}$ $j = 2, 3, 4, 5$ and the first derivative of (8) at $x = x_n$ yields the following discrete Schemes.

$$
\begin{aligned}
& y_{n+5} - 5y_{n+1} + 4y_n = \frac{h^2}{24}[2f_{n+5} + 23f_{n+4} + 52f_{n+3} + 66f_{n+:} \\
& \qquad 90f_{n+1} + 7f_n] \\
& y_{n+} \quad y_{n+4} - 4y_{n+1} + 3y_n \\
& \qquad = \frac{h^2}{120}[f_{n+5} + 2f_{n+4} + 142f_{n+3} + 212f_{n+2} + 337f_{n+1} + 26f_n] \\
& y_{n+3} - 3y_{n+1} + 2y_n = \frac{h^2}{240}[2f_{n+5} - 13f_{n+4} + 52f_{n+3} + 202f \\
& \qquad 442f_{n+1} + 35f_n] \\
& y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{240}[f_{n+5} - 6f_{n+4} + 14f_{n+3} + 4f_{n+2} + \\
& \qquad 209f_{n+1} + 18f_n] \\
& z'_n = \frac{1}{10080h}[10080y_n - 10080y_{n+1} + 2462h^2 f_n + 4315h^2 \\
& 3044h^2 f_{n+2} + 1882h^2 f_{n+3} - 682h^2 f_{n+4} + 107h^2 f_{n+5}]
\end{aligned}
\tag{9}
$$

The equation (9) is the proposed block Schemes from the Continuous formulation (8). The Order is

$[6, 6, 6, 6, 6,]^T$ with the error constants

$$C_{P+2} = [-\frac{95}{6048}, -\frac{19}{2016}, \frac{137}{2016}, -\frac{221}{60480}]^T.$$

Also the first derivative of equation (8), were evaluated at

$x = x_{n+j}\ j = 1, 2, (k)$ which yields the following discrete schemes.

$$z'_{n+1} = \frac{1}{10080h}[10080y_n + 10080y_{n+1} + 863h^2f_n + 5674h^2 \\ 2542h^2f_{n+2} + 1492h^2f_{n+3} - 529h^2f_{n+4} + 82h^2f_{n+5}]$$

Order = 6 , Error Constant = $-\frac{731}{12}$ **(10)**

$$z'_{n+2} = \\ \frac{-1}{10080h}[10080y_n + 10080y_{n+1} - 674h^2f_n - 10133h^2f_{n+1} - \\ 4612h^2f_{n+2} + 314h^2f_{n+3} - 10h^2f_{n+4} - 5h^2f_{n+5}]$$ **(11)**

Order = 6 , Error Constant = $\frac{63}{4}$

$$z'_{n+3} = \\ \frac{1}{10080h}[-10080y_n + 10080y_{n+1} + 751h^2f_n + 9482h^2f_{n+1} + \\ 10226h^2f_{n+2} + 5300h^2f_{n+3} - 641h^2f_{n+4} + 82h^2f_{n+5}]$$ **(12)**

Order = 6 , Error Constant = $\frac{-571}{12}$

$$z'_{n+4} = \\ \frac{1}{10080h}[-10080y_n + 10080y_{n+1} + 674h^2f_n + 10021h^2f_{n+1} \\ 8420h^2f_{n+2} + 1245h^2f_{n+3} + 3818h^2f_{n+4} - 107h^2f_{n+5}]$$ **(13)**

Order = 6 , Error Constant = $\frac{-29}{12}$

$$z'_{n+5} = \frac{1}{10080h}[-10080y_n + 10080y_{n+1} + 863h^2f_n + 8810h^2f_{n+1} + 11794h^2f_{n+2} + 686h^2f_{n+3} + 13807h^2f_{n+4} + 3218h^2f_{n+5}] \quad (14)$$

Order = 6 , Error Constant = $\frac{-585}{4}$

IMPLEMENTATION STRATEGIES

Equations (10) through (14) were substituted into Equation (9) and when solved, simultaneously provides for:

$$y_i \;\; i = 1, 2, (k)$$

at once without recourse to any Predictors as was done in Adesanya et al. (2008) and Awoyemi (1999) for providing for,

$$y_i \;, i = 1, 2, (k-1)$$

in the main Method. Hence this is an improvement over these reported works and as such the new block method proposed we expect to gain in terms of efficiency, accuracy and cost effectiveness. Though the advancement of integrations is done sequentially, the results still compare favorably with the exact solution.

NUMERICAL EXPERIMENTS

PROBLEM 1

$$y'' + (6/x)y' + (4/x^2)y = 0 \qquad x > 0$$

$$y(1) = 1, y'(1) = 1 \qquad h = \frac{0.1}{32}$$

Analytic solution is:

$$y(x) = \frac{5}{3x} - 2/(3x^4)$$

PROBLEM 2

$$y'' = x(y')^2 \;, \quad y(0) = 1, y'(0) = \frac{1}{2}, h = \frac{1}{30}$$

Analytic solution is:

$$y''(x) = 1 + \frac{1}{2}In(\frac{2+x}{2-x}) \quad or\ arctanh(\frac{1}{2}x) + 1$$

CONCLUSIONS

We want to conclude that our new block method of uniform Order 6 is suitable for direct solution of general second order differential equations. The block methods are Self Starting and all the discrete schemes used were gotten from a single continuous formulation [CF] and its derivative which are of uniform order of accuracy. The results were obtained in a Block form which speeds up the computation process and the results obtained from the two problems of linear and non linear modes solved converges with the exact solutions.

APPENDIX

Table of results and absolute errors of Problem 1, $(h = \frac{0.1}{32})$

x	Theoretical sol	New block method	Absolute errors
3.125X10^{-3}	1.003076526	1.003114880	3.8354 E-05
6.25 X10^{-3}	1.006057503	1.006132507	7.5004E-05
9.375 X10^{-3}	1.008944995	1.009050915	1.0592 E-04
1.25 X10^{-2}	1.011741018	1.011876494	1.35476 E-04
1.5625 X10^{-2}	1.014447543	1.014603110	1.55567E-04
1.875 X10^{-2}	1.017066494	1.017252866	1.86372E-04
2.1875 X10^{-2}	1.019599755	1.019795810	1.96055E-04
2.5 X10^{-2}	1.022049164	1.022270209	2.21045E-04
2.8125 X10^{-2}	1.024416519	1.024622147	2.05628E-04
3.125 X10^{-2}	1.026703578	1.026981486	2.77908E-04

Table of results and absolute errors of Problem 2 , $(h = \frac{1}{30})$.

x	Theoretical sol	New block method	Absolute errors
0.0333333	1.01666821	1.016668210	---------------
0.1	1.050041729	1.050047620	5.891E-06
0.2	1.100335348	1.100417747	8.2399E-05
0.3	1.151140436	1.151486857	3.46421E-04
0.4	1.202732554	1.203484655	7.52101E-04
0.5	1.255412817	1.25679310	1.380283E-03

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