

# On Some Difference Sequence Spaces.

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## ABSTRACT

In this article we define some difference sequence spaces using a new difference operator. We show that these spaces can be made *BK*-spaces under a suitable norm. We also find their isometrically isomorphic spaces and thus we find the dual of some of the spaces. Furthermore we study the spaces for separable space, reflexive space, Hilbert space and investigate for solid space, symmetric space, monotone space, convergence free and 1-convex space.

(Keywords: difference sequence space, completeness, BK-space, separable space, reflexive space, Hilbert space, basis, continuous dual, solid space, symmetric space.)

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## INTRODUCTION

Let  $w$  denote the space of all scalar sequences and any subspace of  $w$  is called a sequence space. Let  $\ell_\infty$ ,  $c$  and  $c_0$  be the spaces of *bounded*, *convergent* and *null* sequences  $x = (x_k)$  with complex terms, respectively, normed by:

$$\|x\|_\infty = \sup_k |x_k|.$$

The notion of difference sequence spaces was introduced by Kizmaz [1]. It was generalized by Tripathy and Esi [4] as follows:

Let  $m$  be a fixed positive integer. Then,

$$Z(\Delta_m) = \{x = (x_k) \in w : \Delta_m x \in Z\},$$

for  $Z = \ell_\infty$ ,  $c$  and  $c_0$ ,

where  $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$ , for all  $k \in N$ .

For  $m = 1$ , these spaces become  $\ell_\infty(\Delta)$ ,  $c(\Delta)$  and  $c_0(\Delta)$  introduced by Kizmaz [1].

Tripathy and Esi [4] proved that the spaces  $Z(\Delta_m)$ , for  $Z = \ell_\infty$ ,  $c$  and  $c_0$  are Banach spaces normed by:

$$\|x\|_{\Delta_m} = \sum_{r=1}^m |x_r| + \sup_k |\Delta_m x_k|$$

A sequence space  $E$  is said to be *solid* (or *normal*) if  $(x_k) \in E$  implies  $(\alpha_k x_k) \in E$  for all sequences of scalars  $(\alpha_k)$  with  $|\alpha_k| \leq 1$  for all  $k \in N$ .

A sequence space  $E$  is said to be *monotone* if it contains the canonical pre-images of all its step spaces.

A sequence space  $E$  is said to be *symmetric* if  $(x_{\pi(k)}) \in E$  whenever  $(x_k) \in E$ , where  $\pi$  is a permutation of  $N$ .

A sequence space  $E$  is said to be *convergence free* if  $(y_k) \in E$  whenever  $(x_k) \in E$  and  $y_k = 0$  whenever  $x_k = 0$ .

For  $r > 0$ , a non-empty subset  $V$  of a linear space is said to be *absolutely  $r$ -convex* if  $x, y \in V$  and  $|\lambda|^r + |\mu|^r \leq 1$  together imply that  $\lambda x + \mu y \in V$ . A linear topological space  $X$  is said to be  *$r$ -convex* if every neighborhood of  $\theta \in X$  contains an absolutely  $r$ -convex neighborhood of  $\theta \in X$  (see for instance Maddox and Roles [3]).

A norm  $\|\cdot\|$  on a vector space  $X$  is said to be equivalent to a norm  $\|\cdot\|_0$  on  $X$  if there are

positive numbers  $A$  and  $B$  such that for all for all  $x \in X$  we have:

$$A\|x\|_0 \leq \|x\| \leq B\|x\|_0.$$

This concept is motivated by the fact that equivalent norms on  $X$  define the same topology for  $X$ .

An isomorphism of a normed space  $X$  onto a normed space  $Y$  is a bijective linear operator  $T: X \rightarrow Y$  which preserves the norm, that is, for all  $x \in X$ ,

$$\|Tx\| = \|x\|. \text{ (Hence } T \text{ is isometric)}$$

$X$  is then called isomorphic with  $Y$ , and  $X$  and  $Y$  are called isomorphic normed spaces.

Let  $m$  be a non-negative integer. Then we define the following difference sequence spaces using a new difference operator:

$$Z(\Delta_{(m)}) = \{x = (x_k) \in W : \Delta_{(m)}x \in Z\},$$

for  $Z = \ell_\infty, c$  and  $c_0$ , where,

$$\Delta_{(m)}x = (\Delta_{(m)}x_k) = (x_k - x_{k-m}), \text{ for all } k \in N.$$

## MAIN RESULTS

In this section we study the spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  for some linear topological structures.

**Proposition 1.** *The spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are linear.*

**Proof.** Proof is a routine verification and thus omitted.

**Proposition 2.** *The spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are normed linear spaces, normed by:*

$$\|x\|_{\Delta_{(m)}} = \sup_k |\Delta_{(m)}x_k| \quad (1)$$

**Proof.** For  $x = \theta$ , we have  $\|x\|_{\Delta_{(m)}} = 0$ . Conversely,

let  $\|x\|_{\Delta_{(m)}} = 0$ . Then using (2.1), we have,

$$\sup_k |\Delta_{(m)}x_k| = 0$$

This implies that  $\Delta_{(m)}x_k = 0$  for all  $k \geq 1$ .

Considering  $k = 1, \dots, m$ , we get  $x_k = 0$ , by taking  $x_{1-m}, x_{2-m}, \dots, x_0 = 0$ .

Next let  $k = m+1$ , then  $\Delta_{(m)}x_{1+m} = x_{1+m} - x_1 = 0$ . So  $x_{1+m} = 0$ . Proceeding in this way we get  $x_k = 0$  for all  $k \geq 1$ . Hence  $x = \theta$ .

Also,

$$\begin{aligned} \|x + y\|_{\Delta_{(m)}} &= \sup_k |\Delta_{(m)}(x_k + y_k)| \\ &\leq \sup_k |\Delta_{(m)}x_k| + \sup_k |\Delta_{(m)}y_k| \\ &= \|x\|_{\Delta_{(m)}} + \|y\|_{\Delta_{(m)}}. \end{aligned}$$

$$\text{Thus } \|x + y\|_{\Delta_{(m)}} \leq \|x\|_{\Delta_{(m)}} + \|y\|_{\Delta_{(m)}}$$

Finally, for any scalar  $\lambda$

$$\begin{aligned} \|\lambda x\|_{\Delta_{(m)}} &= \sup_k |\Delta_{(m)}(\lambda x_k)| \\ &= |\lambda| \sup_k |\Delta_{(m)}x_k| \\ &= |\lambda| \|x\|_{\Delta_{(m)}}. \end{aligned}$$

This completes the proof.

**Proposition 3.**

(i)  $\ell_\infty(\Delta_{(m)}) \subset c(\Delta_{(m)}) \subset c_0(\Delta_{(m)})$  and the inclusions are proper.

(ii)  $Z \subset Z(\Delta_{(m)})$ , for  $Z = \ell_\infty, c, c_0$  and the inclusions are proper.

**Proof.** Proof is trivial.

**Theorem 2.** The spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are Banach spaces under the norm (1).

**Proof.** Let  $(x^j)$  be a Cauchy sequence in  $\ell_\infty(\Delta_{(m)})$ , where  $(x^j) = (x_k^j) = (x_1^j, x_2^j, \dots)$  for each  $j \geq 1$ . Then for a given  $\varepsilon > 0$ , there exist a positive integer  $n_0$  such that:

$$\|x^i - x^j\|_{\Delta_{(m)}} = \sup_k |\Delta_{(m)}(x_k^i - x_k^j)| < \varepsilon \text{ for all } i, j \geq n_0$$

This implies that  $|\Delta_{(m)}(x_k^i - x_k^j)| < \varepsilon$  for all  $i, j \geq n_0$  and for all  $k \geq 1$ .

This implies that  $(\Delta_{(m)}x_k^i)$  is a Cauchy sequence in  $\mathbb{C}$  for all  $k \geq 1$  and so it is convergent in  $\mathbb{C}$  for all  $k \geq 1$ .

Let  $\lim_{i \rightarrow \infty} \Delta_{(m)}x_k^i = y_k$ , say for each  $k \geq 1$ . Considering  $k=1, 2, \dots$ , we can easily conclude that  $\lim_{i \rightarrow \infty} x_k^i = x_k$ , exists for each  $k \geq 1$ .

Now we can find that for all  $i, j \geq n_0$

$$\sup_k |\Delta_{(m)}(x_k^i - x_k^j)| < \varepsilon$$

Then we have  $\sup_k |\Delta_{(m)}(x_k^i - \lim_{j \rightarrow \infty} x_k^j)| < \varepsilon$  for all  $i \geq n_0$

Hence,

$$\sup_k |\Delta_{(m)}(x_k^i - x_k)| < \varepsilon \text{ for all } i \geq n_0.$$

This implies that  $(x^j - x) \in \ell_\infty(\Delta_{(m)})$ . Also  $\ell_\infty(\Delta_{(m)})$  is linear.

Hence  $x = x^j - (x^j - x) \in \ell_\infty(\Delta_{(m)})$ .

Thus  $\ell_\infty(\Delta_{(m)})$  is complete.

Similarly it can be shown that the spaces  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are complete.

From the above proof we can easily conclude that  $\|x^i - x\|_{\Delta_{(m)}} \rightarrow 0$  implies  $|x_k^i - x_k| \rightarrow 0$  as  $i \rightarrow \infty$ , for each  $k \geq 1$ . Hence we have the following proposition.

**Proposition 5.** The spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are BK-spaces.

**Remark 1.** From the definition of the operator  $\Delta_m$  and  $\Delta_{(m)}$ , it is clear that for  $Z = \ell_\infty, c$  and  $c_0$ ,  $(\Delta_{(m)}x_k) \in Z$  if and only if  $(\Delta_m x_k) \in Z$ . Hence it is obvious that the norms  $\|\cdot\|_{\Delta_m}$  and  $\|\cdot\|_{\Delta_{(m)}}$  are equivalent.

**Proposition 2.6.** The spaces  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are nowhere dense subsets of  $\ell_\infty(\Delta_{(m)})$ .

**Proof.** From Proposition 3 (i) it follows that the inclusions  $c(\Delta_{(m)}) \subset \ell_\infty(\Delta_{(m)})$  and  $c_0(\Delta_{(m)}) \subset \ell_\infty(\Delta_{(m)})$  are strict. Further from Theorem 2.4, it follows that the spaces  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are closed. Hence the spaces  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are nowhere dense subsets of  $\ell_\infty(\Delta_{(m)})$ .

**Proposition 2.** The spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are isometrically isomorphic to the spaces  $\ell_\infty, c$  and  $c_0$ , respectively.

**Proof.** For  $Z = \ell_\infty, c$  and  $c_0$ , consider the mapping  $T: Z(\Delta_{(m)}) \rightarrow Z$ , defined by:

$$Tx = y = (\Delta_{(m)}x_k), \text{ for every } x \in Z(\Delta_{(m)}).$$

Then clearly  $T$  is linear and  $\|Tx\| = \sup_k |y_k| = \sup_k |(\Delta_{(m)}x_k)| = \|x\|$ .

Hence  $T$  is an isometry.

Also  $T$  is one-one, since  $\ker T = \{0\}$ .

To show  $T$  is on-to, let  $z = (z_k) \in Z$ . Consider the sequence  $x = (x_k) = (z_1, \dots, z_m, z_{m+1} + z_1, z_{m+2} + z_2, \dots)$ . Then  $x \in Z(\Delta_{(m)})$  and  $Tx = z$ .

Thus  $T$  is an isometric isomorphism of  $Z(\Delta_{(m)})$  on-to  $Z$ .

**Theorem 8.** *The continuous dual of  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  is  $\ell_1$ .*

**Proof.** Proof follows from the fact that continuous dual of  $c$  and  $c_0$  is  $\ell_1$  and  $c(\Delta_{(m)})$ ,  $c_0(\Delta_{(m)})$  are isometrically isomorphic to  $c$  and  $c_0$ , respectively.

**Theorem 9.** *The spaces  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are separable.*

**Proof.** Since  $\ell_1$  is separable, the proof follows from the fact that if the dual of a normed space is separable, then the space itself is separable.

**Theorem 10.** (i) *The spaces  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are not reflexive.*

(ii) *The spaces  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are not Hilbert space.*

(iii) *The space  $\ell_\infty(\Delta_{(m)})$  is not a Hilbert space.*

(iv) *The space  $\ell_\infty(\Delta_{(m)})$  is not reflexive.*

**Proof.** (i) Since  $\ell_1$  is not reflexive, the proof follows from the fact that if a normed space is reflexive then its dual is also reflexive.

(ii) Proof follows from the fact that every Hilbert space is reflexive.

(iii) We know that a closed subspace of a Hilbert space is Hilbert space. Here  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are closed subspaces of  $\ell_\infty(\Delta_{(m)})$  but both of them are not Hilbert space. So  $\ell_\infty(\Delta_{(m)})$  is not a Hilbert space.

(iv) We know that a closed subspace of a reflexive Banach space is reflexive. Here  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are closed subspaces of  $\ell_\infty(\Delta_{(m)})$  but both of them are not reflexive. So  $\ell_\infty(\Delta_{(m)})$  is not reflexive.

**Theorem 11.** (i) *The space  $\ell_\infty(\Delta_{(m)})$  is not separable.*

(ii) *The space  $\ell_\infty(\Delta_{(m)})$  has no Schauder basis (basis).*

**Proof.** (i) We can associate for every  $y' \in [0,1]$ , a sequence  $y = (y_i) \in \ell_\infty(\Delta_{(m)})$  of zeros and ones,

where  $y' = \frac{y_1}{2} + \frac{y_2}{2^2} + \frac{y_3}{2^3} + \dots$ . Since  $[0,1]$  is uncountable, so there are uncountably many sequences of zeros and ones. For any two different sequences  $x$  and  $y$  of  $\ell_\infty(\Delta_{(m)})$  we

have  $\|x - y\|_{\Delta_{(m)}} = \sup_k |\Delta_{(m)}(x_k - y_k)| = 1$ .

If we let each of these sequences be the centers of neighborhoods, say, of radius  $1/3$ , these neighborhoods do not intersect and we have uncountably in many of them. If  $D$  is any dense set in  $\ell_\infty(\Delta_{(m)})$ , each of these non intersecting neighborhoods must contain an element of  $D$ .

Hence  $D$  can not be countable. Since  $D$  was an arbitrary dense set, this shows that  $\ell_\infty(\Delta_{(m)})$  can not have countable dense subset. Consequently,  $\ell_\infty(\Delta_{(m)})$  is not separable.

(ii) Proof follows from the fact that if a space has a basis then it is separable.

**Theorem 12.** (i) *The spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are not solid spaces.*

(ii) *The spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  (for  $m > 1$ ) are not symmetric spaces.*

(iii) *The spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are not convergence free.*

(iv) The spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are not monotone.

(v) The spaces  $\ell_\infty(\Delta_{(m)})$ ,  $c(\Delta_{(m)})$  and  $c_0(\Delta_{(m)})$  are 1-convex.

**Proof.** We can find same results (i) to (v) for the spaces  $\ell_\infty(\Delta_m)$ ,  $c(\Delta_m)$  and  $c_0(\Delta_m)$  in Tripathy and Esi [4]. Hence the results (i) to (v) follow by remark 1.

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