Improved Parallel Methods for Second Order ODEs.

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ABSTRACT

Improved zero stable block method of order $(4,4,4)^T$ are proposed for second order initial / boundary value problem of the type $y'' = f(x, y)$, $y(0)$, $y'(0)$ *or* $y'' = f(x, y)$, $y(a)$, $y(b)$ is given. In Fatunla (1992 abd 1994), the author proposed non-uniform Block method in parallel mode for our class of problem in a predictive-corrector mode, but do not include any numerical examples. This motivates the present paper. Novelty of this improvement is to show that some continuous finite difference (FD) formulae can be used to provide a uniform treatment of both the IVP and the BVP. Some computed results are given to show the effectiveness of the proposed method.

(Keywords: multistep, collocation, second order initial/boundary valued problems, block, parallel, zero stability)

INTRODUCTION

In this paper the numerical solution of the second order IVP or BVP:

$$y'' = f(x, y)\,,\ y(0) = y_0\ ,\ y'(0) = \zeta_0 \qquad \textbf{(1)}$$

$$y'' = f(x, y)\,,\ y(a) = \eta\ ,\ y'(b) = \beta \qquad \textbf{(2)}$$

systems (1) occur in mechanical system without dissipation, satellite tracking and celestial mechanics whose theoretical solutions are usually highly oscillatory and thus severally restrict the mesh sizes of the conventional linear multistep methods:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \qquad \textbf{(3)}$$

or compactly of the form:

$$\rho(E) y_n = h^2 \sigma(E) f_n \qquad \textbf{(4)}$$

where E ' s the shift operator specified by

$$E^j y_n = y_{n+j} \qquad \textbf{(5)}$$

while ρ *and* σ are characteristic polynomials and are given as:

$$\rho(r) = \sum_{j=0}^{k} \alpha_j r^j \ ,\ \delta(r) = \sum_{j=0}^{k} \beta_j r^j \qquad \textbf{(6)}$$

y_n is the numerical approximation to the theoretical solution $y(x_n)$ and $f_n = f(x_n, y_n)$. Jeltsch(1978) gave a complete characterization of method (3) with respect to Henrici (1962), Lambert and Watson(1976), Hairer (1976) and Dahlquist (1978).

In this paper, single -Block r-point methods is proposed for the second order ODE's type (1 and 2) in the spirit of Fatunla(1992;1994), Chu and Hamilton(1987) where r is the number of processors and the solution to (1 and 2) at r points are generated simultaneously within any specified block. It is thus possible to assign the computational tasks at r points within the block to r different processors.

THEORY OF BLOCK METHODS FOR SECOND ORDER IVPS

Within the r-vector Y_m and F_m (for n=mr, m=0,1,..)

$$Y_m = (y_{n+1}, y_{n+2}, ..., y_{n+r})^T \ ,\ F_m = (f_{n+1}, f_{n+2}, ..., f_{n+r})^T \qquad \textbf{(7)}$$

The s-block r-point methods for (3) are given by the matrix finite difference equation:

$$-A^{(0)}Y_m = \sum_{i=0}^{s} A^{(i)}Y_{m-i} + h^2 \sum_{i-0}^{s} B^{(i)}F_{m-i} \quad \textbf{(8)}$$

where $A^{(i)}$, $B^{(i)}$, $i = 0(1)s$ *are* $r \times r$ matrices, respectively, with element $a_{ij}^{(i)}$, $b_{ij}^{(i)}$ *for* $i, j = 0(1)r$.

The block scheme (8) is explicit if the coefficient matrix $B^{(0)}$ is a null matrix. Let

$$Z_m = \begin{pmatrix} y(x_{n+1}) \\ y(x_{n+2}) \\ y(x_{n+3}) \\ . \\ . \\ . \\ y(x_{n+r}) \end{pmatrix} \quad \textbf{(9)}$$

represent the theoretical solution to (1 & 2).

Definition 1: The defect or local truncation error (l.t.e.) of the block method (8) is given by the vector E_m:

$$E_m = Z_m - \sum_{i=1}^{s} A^{(i)}Z_{m-i} - h^2 \sum_{i=0}^{s} B^{(i)}Z_{m-1} \quad \textbf{(10)}$$

Definition 2: The block method (8) has error order p>=1 provided there is a constant C such that the defect E_m satisfies

$$\|E_m\| = Ch^{p+2} + O(h^{p+3}) \quad \textbf{(11)}$$

where $\|.\|$ is a suitable norm.

Definition 3: The block method (8) is zero stable provided the root λ_j, $j = 1(1)s$ of the first characteristic polynomial $\rho(\lambda)$ specified as,

$$\rho(\lambda) = \det\left[\sum_{i=0}^{s} A^{(i)}\lambda^{s-1}\right] = 0 \quad \textbf{(12)}$$

satisfies $|\lambda_j| \le 1$, and for those roots with $|\lambda_j| = 1$ multiplicity must not exceed 2.

The principal root of $\rho(\lambda)$ is denoted by $\lambda_1 = \lambda_2 = 1$.

Definition 4: We consider the linear operator L given as:

$$L[Z(x), h] = \sum_{j=0}^{\infty}\left[\alpha_j Z(x+jh) - h^2\beta_j Z''(x+jh)\right] \quad \textbf{(13)}$$

where $Z(x)$ is the exact solution to (1) and assume to be sufficiently differentiable. We now invoke the Taylor's theorem in (13) to obtain:

$$L[Z(x), h] = C_0 Z(x) + C_1 hZ'(x) + ... + C_q h^q Z^{(q)}(x) + O(h^{q+1}) \quad \textbf{(14)}$$

whose coefficient C_q, $q = 0,1,...$ are constants independent of $Z(x)$ and are given as:

$$\left(\begin{array}{c} C_0 = \sum_{j=0}^{k} \alpha_j \\ C_0 = \sum_{j=1}^{k} j\alpha_j \\ . \\ . \\ . \\ C_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q \alpha_j - q(q-1)\sum_{j=1}^{k} j^{r-2}\beta_j\right] \end{array}\right) \quad \textbf{(15)}$$

The order of p of the difference operator $L[Z(x), h]$ is unique integer p such that $C_q = 0$, $q = 0(1)p+1$, $C_{p+2} \neq 0$.

DERIVATION OF 2-POINT BLOCK METHODS

The 1-block 2-point optimal method for second order IVPs, type (1) can be describe by the matrix difference equation,

$$-A^{(0)}Y_m = \sum_{i=0}^{s} A^{(1)}Y_{m-1} + h^2\left[B^{(0)}F_m + B^{(1)}F_{m-1}\right] \quad (16)$$

where,

$$Y_m = \left(y_{n+1}\ ,\ y_{n+2}\right)^T \equiv \left(y_{n+2}\ ,\ y_{n+3}\right)^T$$
$$Y_{m-1} = \left(y_{n-1}\ ,\ y_n\right)^T \equiv \left(y_n\ ,\ y_{n+1}\right)^T$$
$$f_m = \left(f_{n-1}\ ,\ f_n\right)^T \equiv \left(f_n\ ,\ f_{n+1}\right)^T$$
$$f_m = \left(f_{n+1}\ ,\ f_{n+2}\right)^T \equiv \left(f_{n+2}\ ,\ f_{n+3}\right)^T$$

We then consider the following parameter specifications;

$p_i(x) = x^i$, $i = 0,1,2,3,4,5$; $k = 3$, $t = 2$, $m = 4$, $\{x_n\ ,\ x_{n+1}\}$ as the interpolation point; $\{x_n\ ,\ x_{n+1}\ ,\ x_{n+2}\ ,\ x_{n+3}\}$ as the collocation points.

$$\underline{V} = \left(y_n\ ,\ y_{n+1}\ ,\ f_n\ ,\ f_{n+1}\ ,\ f_{n+2}\ ,\ f_{n+3}\right)^T ;$$

$$\underline{\underline{M}} = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 \\ 0 & 0 & 2 & 6x_{n+3} & 12x_{n+3}^2 & 20x_{n+3}^3 \end{pmatrix} \quad (17)$$

$$\underline{\underline{C}} = \begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} & C_{1,5} & C_{1,6} \\ C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} & C_{2,5} & C_{2,6} \\ C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} & C_{3,5} & C_{3,6} \\ C_{4,1} & C_{4,2} & C_{4,3} & C_{4,4} & C_{4,5} & C_{4,6} \\ C_{5,1} & C_{5,2} & C_{5,3} & C_{5,4} & C_{5,5} & C_{5,6} \\ C_{6,1} & C_{6,2} & C_{6,3} & C_{6,4} & C_{6,5} & C_{6,6} \end{pmatrix} \quad (18)$$

with $y(x)$ taking the form

$$y(x) = \sum_{i=0}^{5}\left[\sum_{j=0}^{1} C_{i+1,j+1}y_{n+j} + \sum_{j=0}^{3} C_{i+1,j+3}f_{n+j}\right]x^j \quad (19)$$

Substituting values of C_5 in (19) and simplifying the resulting expression yields the continuous formulation of the method as:

$$\begin{aligned} y(x) &= \left[\frac{-(x-x_{n+1})}{h}\right]y_n + \left[\frac{h+(x-x_{n+1})}{h}\right]y_{n+1} \\ &+ \left[\frac{-3(x-x_{n+1})^5 + 15h(x-x_{n+1})^4 - 20h^2(x-x_{n+1})^3 + 38h^4(x-x_{n+1})}{360h^3}\right]f_n \\ &+ \left[\frac{3(x-x_{n+1})^5 - 10h(x-x_{n+1})^4 - 10h^2(x-x_{n+1})^3 + 60h^3(x-x_{n+1})^2 + 57h^4(x-x_{n+1})}{120h^3}\right]f_{n+1} \\ &+ \left[\frac{-3(x-x_{n+1})^5 + 5h(x-x_{n+1})^4 - 20h^2(x-x_{n+1})^3 + 12h^4(x-x_{n+1})}{120h^3}\right]f_{n+2} \\ &+ \left[\frac{3(x-x_{n+1})^5 + 10h^2(x-x_{n+1})^3 + 7h^4(x-x_{n+1})^3 + 38h^4(x-x_{n+1})}{360h^3}\right]f_{n+3} \end{aligned} \quad (20)$$

SPECIAL APPLICATION OF (20) CONTINUOUS SOLUTION FORM

1. Evaluation of equation (20) at $x = x_{n+2}$ *and* $x = x_{n+3}$ yields respectively the two integrators below

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{12}\{ f_{n+2} + 10 f_{n+1} + f_n \} \ , \ C_6 = -\tfrac{1}{240},$$

$$y_{n+3} - 3y_{n+1} + 2y_n = \frac{h^2}{12}\{ f_{n+3} + 12 f_{n+2} + 21 f_{n+1} - 2 f_n \} \ , \ C_6 = -\tfrac{1}{80},$$

which in matrix Block form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+2} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} y_n \\ y_{n+1} \end{pmatrix} + h^2 \left[\begin{pmatrix} \frac{1}{12} & \frac{5}{6} \\ \frac{1}{6} & \frac{7}{4} \end{pmatrix}\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} + \begin{pmatrix} \frac{1}{12} & 0 \\ 1 & \frac{1}{12} \end{pmatrix}\begin{pmatrix} f_{n+2} \\ f_{n+3} \end{pmatrix} \right] \quad \textbf{(21)}$$

2. The first integrator in the block (21) is the well celebrated optimal method christened "Numerov Method".

3. The block method (21) was obtained via an approach that predetermined its order and convergence by putting some constrained on the parameters of the matrix difference equation (16) in Fatunla (1992). A major set back inherent in the application of (21) is that it is not self-starting, thus it is recommended for use in a predictor-corrective mode along with an order 3-predictor which undoubtedly will corrupt its accuracy. It is also over single step integration with overlapping sub intervals.

4. The Block Method (21) was improved by considering an additional equation arising from the use of first derivative function

$$\frac{dy(x)}{dx} = Z(x) \ , \ \frac{dy(a)}{dx} = Z_0 \quad \textbf{(22)}$$

and the continuity equation imposed at $x=x_{n+3}$ of the form

$$Z\left(x_{n+3}\right)\Big|_{x_n \le x \le x_{n+3}} = Z\left(x_{n+3}\right)\Big|_{x_{n+3} \le x \le x_{n+3k}} \quad \textbf{(23)}$$

Comment 1: It is worthwhile noting here that this improvement is only possible with our present approach because the block solution scheme is available in a continuous form. See Yahaya (2004).

5. To start the IVP integration process, we use the second condition in equation (22) on equation (20) to obtain:

$$hZ_0 = -y_0 + y_1 + \frac{h^2}{360}\{-97 - 114 f_1 + 39 f_2 - 8 f_3\} \ , \ C_6 = -\frac{7}{480} \quad \textbf{(24)}$$

Similarly, the condition (23) yields the continuity equation:

$$-y_{n+4} + y_{n+3} + y_{n+1} - y_n$$
$$= \frac{h^2}{360}\{-8 f_{n+6} + 39 f_{n+5} - 114 f_{n+4} - 224 f_{n+3} - 444 f_{n+2} - 291 f_{n+1} - 38 f_n\}, \quad \textbf{(25)}$$
$$C_6 = \frac{3}{80}$$

Comments 2: The improve Block Method over Fatunla (1992), now become 1-block-3-point method against its earlier form of 1-block -2-points methods. The new 1-block-3-points method posses the following desirable properties:

- being self starting
- produce solution over sub-intervals that do not overlap
- apply uniformly to both IVPs and BVPs with adjustment to the boundary conditions
- stability at origin
- cheap and reliable error estimates

Hence, the resultant integration scheme has eliminated all the associated drawback of Fatunla (1992). The improved Fatunla Block Method (1-block-3-point Method) is then given by (21) and (24) to start the integration process for IVPs type (1), while the combination of (21) and (25) is used for the rest integration process yielding uniform order four multiple FDMs.

STABILITY OF THE 1-BLOCK 3-POINT METHODS

When (21) and (24) are combined and put in matrix equation form, for this analysis the resulting Block method was normalized to obtain:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}y_{n-2}\\y_{n-1}\\y_{n}\end{pmatrix}+h^2\left[\begin{pmatrix}\frac{317}{1000}&-\frac{27}{250}&\frac{11}{500}\\\frac{1467}{1000}&\frac{133}{1000}&\frac{11}{250}\\\frac{27}{10}&\frac{27}{40}&\frac{3}{20}\end{pmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{pmatrix}+\begin{pmatrix}0&0&\frac{269}{1000}\\0&0&\frac{311}{500}\\0&0&\frac{39}{40}\end{pmatrix}\begin{pmatrix}f_{n-2}\\f_{n-1}\\f_{n}\end{pmatrix}\right] \quad \textbf{(26)}$$

The first characteristic polynomial of the improved block method (26) is:

$$\begin{aligned}\rho(R)&=\det\left[RA^{(0)}-A^{(1)}\right]\\&=\det\left[R\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix}\right]\\&=\det\begin{pmatrix}R&0&-1\\0&R&-1\\0&0&R-1\end{pmatrix}\\&=R^2(R-1)\end{aligned} \quad \textbf{(27)}$$

(26) is consistence as it's order p>1. From definition 3 and equation (27), the block method (26) is also zero stable. From Henrici (1962), we can safely assert the convergence of the block method (26).

NUMERICAL EXPERIMENTS

To test the efficiency of the proposed 1-Block 3-point s Methods, two problems' one Stiff -linear and Non-linear was considered.

Problem 1 (Stiff-linear Odes) ;

$$y'' = -100y \quad , y(0) = 1, \; y'(0) = 10$$

Table of Results 1: h=0.001

X	Exact Value	Approx Value	Present Method Error
0	1.000000000	1.000000000	0.00000E+00
0.001	1.009949830	1.009949194	6.36000E-07
0.002	1.019798670	1.019797395	1.27500E-06
0.003	1.029545530	1.029543616	1.91400E-06

Table of Results 2: h=0.0025

X	Exact Value	Approx Value	Present Method Error
0	1.000000000	1.000000000	0.00000E+00
0.0025	1.024684910	1.024674782	1.01280E-05
0.005	1.048729430	1.048709176	2.02540E-05
0.0075	1.072118530	1.072088159	3.03710E-05

Table of Results 3: h=0.005

X	Exact Value	Approx Value	Present Method Error
0	1.000000000	1.000000000	0.00000E+00
0.005	1.048729430	1.048646728	8.27020E-05
0.01	1.094837580	1.094672387	1.65193E-04
0.015	1.138209210	1.137961933	2.47277E-04

Problem 2 (Non-linear Odes);

$y'' = 2y^3, \; y(1) = 1, \; y'(1) = -1$ Exact solution; $y(x) = \dfrac{1}{x}$

Table of Results 4: h=0.1

N	X	Exact value	Approx value	Error
0	1	1.000000000	1.000000000	0.00000E+00
1	1.1	0.909090109	0.909095654	5.54500E-06
2	1.2	0.833333333	0.833344394	1.10612E-05
3	1.3	0.769230769	0.769248998	1.82287E-05
4	1.4	0.714285714	0.714304838	1.91236E-05
5	1.5	0.656666667	0.666686973	1.00203E-02
6	1.6	0.625000000	0.625021667	2.16667E-05
7	1.7	0.588235294	0.588257194	2.19001E-05
8	1.8	0.555555556	0.555577763	2.22066E-05
9	1.9	0.526315789	0.526338353	2.25640E-05
10	2.0	0.500000000	0.500022639	2.26392E-05

Problem 3 (boundary value problem);

$$y'' = 3x + 4y \quad , y(0) = 0 , \; y(1) = 1$$

Solve the equation in the range [0,1] using h=0.2

The exact solution is
$$y = \frac{7\left(e^{2x} - e^{-2x}\right)}{4\left(e^{2} - e^{-2}\right)} - \frac{3}{4}x$$

Table of Results 5: h=0.2

X	Exact Value	Approx Value	Absolute Error
0	0.00000000000	0.00000000000	0.00000E+00
0.2	0.04819251100	0.04820849015	1.59791E-05
0.4	0.12852089500	0.12855889760	3.80026E-05
0.6	0.27833169000	0.27840534250	7.36525E-05
0.8	0.54623764000	0.54626247100	2.48310E-05
1.0	1.00000000000	1.00000000000	0.00000E+00

CONCLUSION

A Collocation technique which yields a method with Continuous Coefficients has been presented for the approximate Solution of Special Second Order ODEs with uniform treatment for either initial or boundary conditions. Three test examples have been solved to demonstrate the efficiency of the proposed methods and the Results Compare Favorably with the exact Solution ,a desirable feature of a good numerical methods. Interestingly, all the discrete schemes used in the Block formulation were from a single continuous formulation (CF).

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SUGGESTED CITATION

Yahaya, Y.A. 2009. “Improved Parallel Methods for Second Order ODEs”. *Pacific Journal of Science and Technology*. 10(2):235-242.

Pacific Journal of Science and Technology