

On the Use of Characteristic Function for Generating Moments of Probability Distributions.

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ABSTRACT

Characteristic Functions (cf) have been used to establish the convergence of several independent and identically distributed (*i.i.d*) random variables. It is also used in identifying uniquely, the distribution of random variables (i.e., once the (cf) is known), one can categorically state the distribution of the random variable involved. In this paper, the use of the (cf) in generating moments of random variables about origin is presented. Means and variance of the distribution are also obtained. Two discrete and two continuous probability distribution are taken into consideration.

(Keywords: characteristic functions, CF, independent and identically distributed random variables, IID)

INTRODUCTION

Let X be a random variable with density function $f(x)$ and distribution function $F(x)$. The characteristic function $\phi_x(t)$ of the random variable X is defined by:

$$\phi_x(t) = E(e^{itx}) \tag{1}$$

$$= \int_x e^{itx} f(x) dx \tag{2}$$

where i is the imaginary unit, $i = \sqrt{-1}$, and t is real.

Alternatively, $\phi_x(t)$ is frequently referred to as the characteristic function corresponding to the distribution $F(x)$ since, by definition $e^{itx} = \cos tx - i \sin tx$.

It follows that:

$$\left| \phi_x(t) \right| \leq \int_x \left| e^{itx} \right| f(x) dx = 1. \tag{3}$$

This implies that the integral in the definition of $\phi_x(t)$ will exist for any $f(x)$ for all t , and hence that, unlike the moment generating function, the characteristic function of a random variable always exists. It is also of importance to note that:

$$\phi_x(t) \text{ is always finite i.e } \left| \phi_x(t) \right| = 1$$

$$\text{and } \left| \phi_x(t)^{-1} \right| < \infty \tag{4}$$

Properties of Characteristic Function

- a.
 - (i) $\phi_x(t)$ is uniformly continuous on the real line
 - (ii) $\phi_x(0) = 1$
 - (iii) $\left| \phi_x(t) \right| \leq 1 \quad \forall t$. Since $\left| e^{itx} \right| \leq 1$
- b. The Characteristic function of the sum of independent random variables is the product of their characteristic function i.e If $s_n = x_1 + x_2 + x_3 + \dots + x_n$ where x_1, x_2, \dots, x_n are *i.i.d* random variables then:

$$\phi_{s_n}(t) = \phi_{x_1}(t) \phi_{x_2}(t) \dots \phi_{x_n}(t)$$

$$= \left(\phi_x(t) \right)^n$$
- c. Unlike the moment generating functions, $\phi_x(t)$ is finite \forall variables x and all real

number t . This is because e^{it} is bounded while e^t is not bounded for $-\infty < t < \infty$.

- d. The distribution function of x and hence the pdf, if it exists can be obtained from the characteristic function using an "inversion formula". If x is a discrete random variable, then:

$$f_x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \phi_x(t) dx.$$

If x is continuous, then:

$$f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} \phi_x(t) dt \quad \text{assuming} \\ \int_{-\infty}^{\infty} |\phi_x(t)| dt < \infty.$$

- e. If two random variable have the same characteristic function, then they have the same distribution function.

USES OF CHARACTERISTIC FUNCTION

Characteristic function of a random variable $X(w)$ defined on (Ω, A, P) provided a powerful and applicable tool in the theory of probability. Characteristic functions is used to prove both the weal law of large numbers and the central limit Theorem among others.

The Central Limit Theorem

Let $\{x_i\}$ be the sequence of independent and identically distributed random variables each having mean μ and finite non-zero variance σ^2 . Consider the sum.

$$S_n = x_1 + x_2 + \dots + x_n$$

Let the measure $\frac{S_n - n\mu}{\sigma\sqrt{n}}$ be denoted by P_n

Then $P_n \rightarrow N(0, 1)$, i.e P_n converges in measure to the standard normal distribution.

Proof: Let $\phi_x(t)$ be the characteristic function of

$$\text{each } x_i \text{ and let } s_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}} \quad (5)$$

Let $\phi(t)$ be the characteristic function of X By a Taylor (maclaurin) expansion about $t = 0$, we have:

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{t^2}{2}\phi''(0) + 0(t^2) \quad (6)$$

$$= 1 + i\mu t - \frac{(\mu^2 + \sigma^2)}{2}t^2 + 0(t^2) \quad (7)$$

$$\phi_{(s_n^*)}^{(t)} = \phi_x^n(t), \quad s_n^* = \frac{S_n}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma} \quad (8)$$

$$\phi_{s_n^*}^* = e^{(-i\mu\sqrt{n}\frac{t}{\sigma})} \phi^n\left(\frac{t}{\sigma\sqrt{n}}\right) \quad (9)$$

Taking logarithm, we have:

$$\text{Log } \left(\phi_{s_n^*}^{(t)}\right) = -\frac{-i\mu\sqrt{nt}}{\sigma} + n \ln \phi\left(\frac{t}{\sqrt{n}\sigma}\right) \quad (10)$$

$$-\frac{-i\mu\sqrt{nt}}{\sigma} + n \ln \left[1 + \frac{i\mu t}{\sigma\sqrt{n}} - \frac{\mu^2 + \sigma^2}{2n\sigma^2} + 0\left(\frac{t^2}{n\sigma^2}\right) \right] \quad (11)$$

Using the power series:

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots, |z| < 1, \quad (12)$$

we obtain

$$\ln \left(\phi_{s_n^*}^{(t)}\right) = -\frac{i\mu\sqrt{nt}}{\sigma} + \left[\frac{i\mu t}{\sigma\sqrt{n}} - \frac{\mu^2 + \sigma^2}{2n\sigma^2} + \frac{\mu^2 + t^2}{2n\sigma^2} + 0\left(\frac{t^2}{n\sigma^2}\right) \right] \quad (13)$$

$$\ln \left(\phi_{s_n^*}^{(t)}\right) = -\frac{t^2}{2} + n \cdot 0 \left(\frac{t^2}{n\sigma^2}\right) \quad (14)$$

Taking limit $0 \left(\frac{1}{n} \right) / \frac{1}{n} \rightarrow 0$

$$\lim_{n \rightarrow \infty} l_n \left(\phi_{s_n^x}(t) \right) = -\frac{t^2}{2} \quad (15)$$

$$\text{Thus } \phi_{s_n^x}(t) \rightarrow e^{-t^2/2} \quad (16)$$

Which is the characteristic function of a standard normal random variable, i.e.:

$$\lim_{n \rightarrow \infty} l_n \phi_{s_n^x}(t) \rightarrow \phi_{s_1^x}(t), \quad (17)$$

hence $P_n \rightarrow N(0, 1)$ (Q.E.D)

Let t be fixed, if $t \neq 0$, we have

$$\lim_{n \rightarrow \infty} n \left\{ \log \phi_x \left(\frac{t}{n} \right) - \frac{i\mu t}{n} \right\} = t \lim_{n \rightarrow \infty} \frac{\log \phi_x \left(\frac{t}{n} \right) - i\mu \left(\frac{t}{n} \right)}{t/n} \quad (20)$$

$$\text{But } \lim_{n \rightarrow \infty} \log \frac{\phi_x \left(\frac{t}{n} \right)}{t/n} = i\mu. \text{ If } t=0, \text{ It } t=0, \phi(0) = 1 \quad (21)$$

$$\text{Thus } \lim_{n \rightarrow \infty} n \left\{ \log \phi_x \left(\frac{t}{n} \right) - \frac{i\mu t}{n} \right\} = 0 \quad \forall t. \quad (22)$$

Therefore, $\phi_{s_n^*}(t) \rightarrow 1$ as $n \rightarrow \infty$

$$\text{That is, } \phi_x(t) = 1 = E(e^{itx}) \text{ implies that } P(x=0) = 1. \quad (23)$$

Hence, the distribution function of x is given by:

$$F_x(x) = \begin{cases} 1 & f(x) > 0 \\ 1 & f(x) > 0 \end{cases} \quad (24)$$

Weak Law of Large Numbers

Let $\{x_i\}$ be a sequence of independent identically distributed random variables having mean μ .

Let $S_n = x_1 + x_2 + \dots + x_n$

Then $\frac{S_n}{n} \rightarrow \mu$ in measure.

Proof: Let $\phi_x(t)$ be the characteristic function of x_1

$$\text{Let } s_n^* = \frac{S_n}{n} - \mu, \phi_{s_n^*}(t) \rightarrow \phi_{s_1^*}(t) \quad (18)$$

$$= \exp \left\{ n \left[\log \phi_x \left(\frac{t}{n} \right) - i \frac{\mu t}{n} \right] \right\} \quad (19)$$

This shows that $s_n^* \rightarrow 0$ in measure. By the continuity theorem:

$$\left. \begin{aligned} \lim_{n \rightarrow \infty} P(s_n^* \leq -\epsilon) &= F_x(-\epsilon) = 0. \\ \lim_{n \rightarrow \infty} P(s_n^* \leq \epsilon) &= F_x(\epsilon) = 1. \end{aligned} \right\} \quad (25)$$

Thus $\lim_{n \rightarrow \infty} P(|s_n^*| \leq \epsilon) = 1, \quad \text{Q.E.D} \quad (26)$

That is $\frac{s_n}{n} \rightarrow \mu$ in measure.

THEORETICAL FRAME WORK

To further establish the use of the characteristic function in generating moments of probability distributions just like the other generating functions (probability generating function (pgf), moment generating function (mgf), factorial moment generating function (fmf) etc), the moment, about an arbitrary origin can be obtained for probability distributions, by using the formula divided below.

$$\mu_r^1 = (-i)^r \Phi_x^{(r)}(0) \quad (27)$$

where μ_r^1 is the r^{th} moment about the origin,

$$\begin{aligned} \phi_x^{(n)}(t) &= \frac{d^n}{dt^n} E(e^{itx}) \\ &= E\left[(ix)^n e^{itx}\right] \\ &= E(i^n x^n) e^{itx} \\ &= i^n E(x^n) e^{itx} \end{aligned}$$

Multiplying both sides by i^n

$$i^n \phi_x^{(n)}(t) = i^{2n} E(x^n) e^{itx}$$

$$= (i^2)^n E(x^n) e^{itx}$$

$$i^n \phi_x^{(n)}(t) = (-1)^n E(x^n) e^{itx}$$

$$(-1)^n i^n \phi_x^{(n)}(t) = E(x^n) e^{itx}$$

$$(-i)^n \phi_x^{(n)}(t) = E(x^n) e^{itx}$$

Let $t = 0$

$$(-i)^n \phi_x^{(n)}(0) = E(x^n) = \mu_r^1$$

$\Phi_x^{(r)}(0)$ is the r^{th} derivative of the characteristic function of the random variable x with respect to t and estimated at $t = 0$.

The distributions that will be considered are: Binomial, Poisson, Gamma and Normal.

Consideration of Two Discrete and Two Continuous Probability Distributions

The Binomial Distribution: The characteristic function is given by the first moment about the

$$\Phi_{(x)}(t) = \left((1-p) + Pe^{it} \right)^n \quad (28)$$

arbitrary origin is

$$\mu_1^1 = (-i) \Phi_x^1(0) \quad (29)$$

which follows from 27

$$\begin{aligned} \Phi_x^1(t) &= n \left((1-p + Pe^{it})^{n-1} pie^{it} \right) \\ &= nPie^{it} (pe^{it} + q)^{n-1} \\ \phi_{(x)}^1(o) &= npie^0 (pe^0 + q) \\ &= inp \end{aligned} \quad (30)$$

Therefore from (29) and (30) we have,

$$\begin{aligned} \mu_1^1 &= (-i)^1 npi \\ &= -inpi = np \end{aligned} \quad (31)$$

The second moment about the origin is similarly obtained as:

$$\mu_2^1 = (-i)^2 \Phi_x^1(0) \quad (32)$$

Equation (32) also follows from (27)

$$\begin{aligned} \phi_x^{11}(t) &= \frac{d}{dt} \phi_x^1(t) \\ &= \frac{d}{dt} \left(nPie^{it} (pe^{it} + q)^{n-1} \right) \\ &= nPie^{it} (n-1) (pe^{it} + q)^{n-2} + n^2 pe^{it} + (pe^{it} + q)^{n-1} (n^2 pe^{it}) \\ &= n(n-1) p_i^2 2 (e^{it})^2 (pe^{it} + q)^{n-2} + n^2 pe^{it} \end{aligned} \quad (33)$$

Setting t to zero, we have:

$$\begin{aligned} \phi_{(x)}^1(o) &= n(n-1)p^2i^2 + (p+q)^{n-2} + n^2p \\ &= -n(n-1)p^2 - np \end{aligned} \quad (34)$$

That is, $\mu_2^1 = (-i)^2 \phi_x^{11}(0)$

$$\begin{aligned} \mu_2^1 &= -n(n-1)p^2 - np \\ \mu_2^1 (-i)^2 \phi_x^{11}(0) &= (-i)^2 (-n(n-1)p^2 - np) \\ &= -1(-n(n-1)p^2 - np) \\ &= np + n(n-1)p^2 \end{aligned} \quad (35)$$

$$\begin{aligned} V(x) &= \mu_2^1 - (\mu_1^1)^2 \\ &= np + n(n-1)p^2 - (np)^2 \\ &= np(1-p) \\ &= np(q) \\ &= npq \end{aligned} \quad (36)$$

Equation (31) and (35) gives, respectively, the 1st and 2nd moments by virtue of the characteristic function approach and consequently (36) called the variance of the Binomial distribution.

The Poisson Distribution

The characteristic function in given by Equation

$$(1) \text{ as } \phi_x(x) = e^\lambda e^{\lambda(e^{it}-1)} \quad (37)$$

Using (27), we have,

$$\phi_x^1(t) = \lambda i e^{it} e^{\lambda(e^{it}-1)} \quad (38)$$

Applying (29), then

$$\begin{aligned} \mu_1^1 &= (-i) \phi_x^1(o) \\ &= (-i) \lambda i (1) e^{\lambda(o)} \\ &= -i^2 \lambda \\ &= \lambda \end{aligned} \quad (39)$$

Similarly, we obtain the 2nd moment using (32), yielding:

$$\begin{aligned} \phi_x^{11}(t) &= \frac{d}{dt} \phi_x^1(t) \\ &= \frac{d}{dt} (\lambda i e^{it} e^{\lambda(e^{it}-1)}) \\ &= \lambda i e^{it} (\lambda i e^{it} e^{\lambda(e^{it}-1)}) + e^{\lambda(e^{it}-1)} (\lambda i^2 e^{it}) \\ &= \lambda^2 i^2 e^{2it} e^{\lambda(e^{it}-1)} + \lambda i^2 e^{it} e^{\lambda(e^{it}-1)} \end{aligned} \quad (40)$$

$$= \lambda^2 i^2 e^{2it} e^{\lambda(e^{it}-1)} + \lambda i^2 e^{it} e^{\lambda(e^{it}-1)} \quad (41)$$

setting t to zero, (41) becomes

$$\phi_x^{11}(0) = \lambda^2 i^2 + \lambda i^2 \quad (42)$$

By similar argument,

$$\begin{aligned} \mu_2^1 &= (-i)^2 \phi_x^{11}(0) \quad \text{from (32)} \\ &= (-i)^2 (\lambda^2 i^2 + \lambda i^2) \\ &= -1 (-\lambda^2 - \lambda) \\ \mu_2^1 &= \lambda^2 + \lambda \end{aligned} \quad (43)$$

$$V(x) = \mu_2^1 - (\mu_1^1)^2 = \lambda^2 + \lambda - (\lambda)^2 \quad (44)$$

$$= \lambda \quad (45)$$

The Gamma Distribution:

$$\phi_x(t) = (1 - \beta it)^{-\alpha}$$

$$\phi_x^1(t) = \alpha \beta i (1 - \beta it)^{-\alpha-1}$$

$$\begin{aligned} \mu_1^1 &= (-i)^1 \phi_x^1(0) \\ &= -\alpha \beta i^2 (1 - 0)^{-\alpha-1} \\ &= \alpha \beta \end{aligned} \quad (47)$$

$$\begin{aligned} \Phi_x^{11}(t) &= \frac{d}{dt} \phi_x^1(t) \\ &= \frac{d}{dt} \alpha \beta i (1 - \beta it)^{-\alpha-1} \end{aligned}$$

$$\Phi_x^{11}(t) = \alpha \beta^2 i^2 (\alpha + 1) (1 - \beta it)^{-\alpha-2} \quad (48)$$

$$\begin{aligned} \mu_2^1 &= (-i)^2 \Phi_x^{11}(0) \\ &= (-i)^2 \phi_x^{11}(0) = \\ &= (-i)^2 \alpha \beta^2 i^2 (\alpha + 1) (1 - 0)^{-\alpha-2} \\ &= i^2 \alpha \beta^2 i^2 (\alpha + 1) \end{aligned}$$

$$\begin{aligned} &= -1 \alpha \beta^2 (-1)(\alpha + 1) \\ &= \alpha \beta^2 (\alpha + 1) \end{aligned} \quad (49)$$

$$\begin{aligned} V(x) &= \mu_2^1 - (\mu_1^1)^2 \\ &= \alpha \beta^2 (\alpha + 1) - (\alpha \beta)^2 \\ &= \alpha \beta^2 \end{aligned} \quad (50)$$

The Normal Distribution:

$$\phi_x(t) = e^{\mu it - \frac{\sigma^2 t^2}{2}} \quad (51)$$

$$\phi_x^1(t) = (i\mu - \sigma^2 t) e^{i\mu t - \frac{\sigma^2 t^2}{2}} \quad (52)$$

$$\mu_1^1 = (-i)(i\mu - \sigma^2(0))e^0 = \mu \quad (53)$$

$$\begin{aligned} \Phi_x^{11}(t) &= \frac{d}{dt} \phi_x^1(t) = (i\mu - \sigma^2 t)^2 e^{i\mu t - \frac{\sigma^2 t^2}{2}} \\ &\quad - \sigma^2 e^{i\mu t - \frac{\sigma^2 t^2}{2}} \end{aligned} \quad (54)$$

$$\begin{aligned} \Phi_x^{11}(0) &= (i\mu)^2 (1) - \sigma^2 \\ &= i^2 \mu^2 - \sigma^2 \\ \Phi_x^{11}(0) &= -\mu^2 - \sigma^2 \end{aligned} \quad (55)$$

$$\begin{aligned} \mu_2^1 &= (-i)^2 (-\mu^2 - \sigma^2) \\ &= \mu^2 + \sigma^2 \end{aligned} \quad (56)$$

$$\begin{aligned} V(x) &= \mu_2^1 - (\mu_1^1)^2 \\ &= \mu^2 + \sigma^2 - (\mu)^2 \\ &= \mu^2 + \sigma^2 - \mu^2 = \sigma^2 \end{aligned} \quad (57)$$

DISCUSSION OF RESULTS

Equations (31) and (35) give, respectively, the first two moments of the Binomial distribution by

virtue of the characteristic function approach presented by (27). This consequently yields (36) called the variance.

Similar operations were subsequently carried out in the poison, Gamma and Normal distributions yielding appropriate results of the first two moments as well as their variances.

It must be stated that the importance of this (characteristic function) approach lies in the fact that it takes care of probability dentition whose moment generating functions do not exist.

CONCLUSION

In view of the foregoing it has been clearly demonstrated and established that apart from obtaining the characteristic functions of probability distributions it can be further used to obtain moments for random variables, be it discrete or continuous, just like other known generating functions.

Since the characteristic function always exist for any random variable x , unlike the moment generating function, it provided the solution to the problem of obtaining important statistical measures like the mean, variances, kurtosis, skewness, *etc.*, for all distributions of random variables

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