Lacunary *I*-Convergent Sequences of Fuzzy Real Numbers.

Bipan Hzaarika

Department of Mathematics, Rajiv Gandhi University, Itanagar- 791 112, Arunachal Pradesh, India.

*E-mail: bh rgu@yahoo.co.in

ABSTRACT

In this article we introduce the concept of lacunary *I*-convergent sequence of fuzzy real numbers and the spaces $(c_\theta^I)_F$ and $((c_0^I)_\theta)_F$. We also obtain some inclusion relations between these spaces.

(Keywords: ideal, *I*-convergent, *I*-Cauchy, fuzzy real number, lacunary sequence)

(AMS Classification No: 40105, 46D05, 46A45)

INTRODUCTION

The works on *I*-convergence of real valued sequences was initially studied by Kostyrko, Šalàt, and Wilczyński [3]. Later it was further studied by Šalàt, Tripathy, and Ziman [6], [7] and many others.

The concept of fuzzy sets was initially introduced by Zadeh [8]. Later on sequences of fuzzy real numbers have been discussed by Nanda [4], Nuray and Savas [5] and many others.

A lacunary sequence is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. The intervals determined by θ and it will be defined by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by ϕ_r.

Freedman, Sember, and Raphael [2] defined the space N_θ in the following way: For any lacunary sequence $\theta = (k_r)$,

$N_\theta = \{(x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} |x_k - L| = 0$, for some $L\}$.

The space N_θ is a BK space with the norm

$$\|(x_k)\|_\theta = \sup_r h_r^{-1} \sum_{k\in I_r} |x_k|.$$

N_θ^0 denotes the subset of these sequences in N_θ for which $\theta = 0$, $(N_\theta^0, \|.\|_\theta)$ is also a BK space.

DEFINITIONS AND NOTATIONS

Let S be a non-empty set. Then a non empty class $I \subseteq 2^S$ (power sets of S) is said to be an *ideal* if I is additive (*i.e.* $A, B \in I \Rightarrow A \cup B \in I$), and hereditary (*i.e.* $A \in I$, $B \subseteq A \Rightarrow B \in I$).

An ideal $I \subseteq 2^S$ is said to be *non trivial* if $I \neq 2^S$.

A fuzzy real number X is a fuzzy set on R, *i.e.* a mapping $X: R \to J$ (= [0, 1]) associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$,

where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

The α-*level* set of a fuzzy real number X, $0 < \alpha \leq 1$ denoted by X^α is defined as:

$X^\alpha = \{t \in R : X(t) \geq \alpha\}$.

A fuzzy real number *X* is said to be *upper semi-continuous* if for each ε > 0, X^{-1}([0, *a*+ε]), for all *a* ∈*J* is open in the usual topology of *R*. The set of all upper semi-continuous, normal, convex fuzzy number is denoted by *R*(*J*).

Let *D* denote the set of all closed and bounded intervals *X* = $[x_1, x_2]$ on the real line *R*. For *X* = $[x_1, x_2]$ and *Y* = $[y_1, y_2]$ in *D*, we define *X* ≤*Y* if and only if $x_1 \le y_1$ and $x_2 \le y_2$. Define a metric *d* on *D* by,

$$d(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

It is known that (*D*, *d*) is a complete metric space and "≤" is a partial order on *D*.

The absolute value |*X*| of *X* ∈*R*(*J*) is defined as

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, \; if \; t > 0; \\ 0, \; if \; t < 0. \end{cases}$$

Let $\bar{d}$: *R*(*J*) × *R*(*J*) → *R* be defined by

$$\bar{d}(X, Y) = \sup_{0 \le \alpha \le 1} d(X^{\alpha}, Y^{\alpha}).$$

Then $\bar{d}$ defines a metric on *R*(*J*).

We define *X* ≤ *Y* if and only if $X^{\alpha} \le Y^{\alpha}$, for all α∈ *J*. The additive identity and multiplicative identity in *R*(*J*) are denoted by $\bar{0}$ and $\bar{1}$, respectively.

A sequence (X_k) of fuzzy real numbers is said to be *convergent* to a fuzzy real number X_0 if for every ε > 0, there exists $n_0 \in N$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \ge n_0$.

A sequence (X_k) of fuzzy real numbers is said to be *I-convergent* if there exists a fuzzy real number X_0 such that for each ε > 0, the set:

$$\{k \in N : \bar{d}(X_k, X_0) \ge \varepsilon\} \in I.$$

We write *I*- $\lim X_k = X_0$.

If *I* = I_f (class of all finite subsets of *N*), then I_f- convergence coincides with the usual convergence of fuzzy numbers.

Let $\theta = (k_r)$ be lacunary sequence. Then a sequence (X_k) of fuzzy real numbers is said to be *lacunary I- convergent* if for every ε > 0 such that:

$$\{r \in N : h_r^{-1} \sum_{k \in I_r} \bar{d}(X_k, X) \ge \varepsilon\} \in I.$$

We write I_θ - $\lim X_k = X$.

Let $\theta = (k_r)$ be lacunary sequence. Then a sequence (X_k) of fuzzy real numbers is said to be *lacunary I - null* if for every ε > 0 such that:

$$\{r \in N : h_r^{-1} \sum_{k \in I_r} \bar{d}(X_k, \bar{0}) \ge \varepsilon\} \in I.$$

We write I_θ - $\lim X_k = \bar{0}$.

Let $\theta = (k_r)$ be lacunary sequence. Then a sequence (X_k) of fuzzy real numbers is said to be *lacunary I-Cauchy* if there exists a subsequence $(X_r^{/}(r))$ of (X_k) such that $k'(r) \in J_r$, for each *r*, $\lim_{r \to \infty} X_k^{/}(r) = X'$ and for every ε > 0 such that:

$$\{r \in N : h_r^{-1} \sum_{k \in J_r} \bar{d}(X_k, X_{k'(r)}) \ge \varepsilon\} \in I.$$

A lacunary sequence $\theta' = (k'(r))$ is said to be a *lacunary refinement* of the lacunary sequence $\theta = (k_r)$ if $(k_r) \subset (k'(r))$.

Throughout w_F, $(\ell_\infty)_F$, $(c_\theta^I)_F$, $((c_0^I)_\theta)_F$ and $(\ell_\infty^I)_F$ denotes *all*, *bounded*, *lacunary I-convergent*, *lacunary I-null* and *I-bounded* class of sequences of fuzzy real numbers, respectively.

MAIN RESULTS

Theorem 1. *A sequence* (X_k) *of fuzzy real numbers is* I_θ*-convergent if and only if* I_θ*-Cauchy sequence*.

Proof. Let $(X_k) \in (c_\theta^I)_F$ be a sequence of fuzzy real numbers. Then there exists $X \in C$ such that $I_\theta - \lim X_k = X$.

Write $H_{(i)} = \{r \in N: h_r^{-1} \sum_{k \in J_r} \bar{d}(X_k, X) < \frac{1}{i}\}$, for each $i \in N$.

Hence for each i, $H_{(i)} \supseteq H_{(i+1)}$ and

$$\{r \in N: h_r^{-1} \sum_{k \in J_r} \bar{d}(X_k, X) < \frac{1}{i}\} \notin I.$$

We choose k_1 such that $r \geq k_1$, then

$$\{r \in N: h_r^{-1} \sum_{k_1 \in J_r} \bar{d}(X_{k_1}, X) < 1\} \notin I.$$

Next we choose $k_2 > k_1$ such that $r \geq k_2$, then

$$\{r \in N: h_r^{-1} \sum_{k_2 \in J_r} \bar{d}(X_{k_2}, X) < \frac{1}{2}\} \notin I.$$

For each r satisfying $k_1 \leq r < k_2$, choose $k'(r) \in J_r$ such that:

$$\{r \in N: h_r^{-1} \sum_{k'(r) \in J_r} \bar{d}(X_{k'(r)}, X) < 1\} \notin I.$$

In general, we choose $k_{p+1} > k_p$, such that $r > k_{p+1}$ then:

$$\{r \in N: h_r^{-1} \sum_{k_{p+1} \in J_r} \bar{d}(X_{k_{p+1}}, X) < \frac{1}{p}\} \notin I.$$

Then for all r satisfying $k_p \leq r < k_{p+1}$, such that

$$\{r \in N: h_r^{-1} \sum_{k'(r) \in J_r} \bar{d}(X_{k'(r)}, X) < \frac{1}{p}\} \notin I.$$

Thus we get $k'(r) \in J_r$, for each r and $\lim_{r \to \infty} X_{k'(r)} = X$.

Therefore, for every ε > 0, we have:

$$\{r \in N: h_r^{-1} \sum_{k, k' \in J_r} \bar{d}(X_k, X_{k'(r)}) \geq \varepsilon\} \subseteq \{r \in N: h_r^{-1} \sum_{k \in J_r} \bar{d}(X_k, X) \geq \frac{\varepsilon}{2}\}$$

$$\cup \{r \in N: h_r^{-1} \sum_{k'(r) \in J_r} \bar{d}(X_{k'(r)}, X) \geq \frac{\varepsilon}{2}\}.$$

i.e. $\{r \in N: h_r^{-1} \sum_{k, k' \in J_r} \bar{d}(X_k, X_{k'(r)}) \geq \varepsilon\} \in I$.

Then (X_k) is a I_θ- Cauchy sequence. Conversely suppose (X_k) is a I_θ-Cauchy sequence. Then for every ε > 0, we have:

$$\{r \in N: h_r^{-1} \sum_{k \in J_r} \bar{d}(X_k, X) \geq \varepsilon\} \subseteq \{r \in N: h_r^{-1} \sum_{k, k' \in J_r} \bar{d}(X_k, X_{k'(r)}) \geq \frac{\varepsilon}{2}\}$$

$$\cup \{r \in N: h_r^{-1} \sum_{k'(r) \in J_r} \bar{d}(X_{k'(r)}, X) \geq \frac{\varepsilon}{2}\}.$$

It follows that (X_k) is a I_θ-convergent sequence.

Theorem 2. *If* θ' *is a lacunary refinement of a lacunary sequence* θ *and* $(X_k) \in (c_{\theta'}^I)_F$, *then* $(X_k) \in (c_\theta^I)_F$.

Proof. Suppose that for each J_r of θ contains the points $(k'_{r,t})_{t=1}^{\eta(r)}$ of θ' such that:

$k_{r-1} < k'_{r,1} < k'_{r,2} < \ldots < k'_{r,\eta(r)} = k_r$, where $J'_{r,t} = (k'_{r,t-1}, k'_{r,t}]$.

Since $k_r \subseteq (k'(r))$, so r, $\eta(r) \geq 1$.

Let $(J_j^*)_{j=1}^{\infty}$ be the sequence of intervals ($J_{r,t}^{/}$) ordered by increasing right end points.

Since (X_k)$\in (c_{\theta'}^I)_F$, then for each $\varepsilon > 0$,

{$j\in N$: $(h_j^*)^{-1} \sum_{J_j^* \subset J_r} \overline{d}(X_k, X) \geq \varepsilon$ }$\in I$.

Also since $h_r = k_r - k_{r-1}$, so $h_{r,t}^{/} = k_{r,t}^{/} - k_{r,t-1}^{/}$.

For each $\varepsilon > 0$, we have:

$$\{r\in N : h_r^{-1} \sum_{k\in J_r} \overline{d}(X_k, X) \geq \varepsilon \} \subseteq$$

$$\{ r\in N \; h_r^{-1} \sum_{k\in J_r} \{ j \in N : (h_j^*)^{-1} \sum_{\substack{J_j^* \subset J_r \\ k\in J_j^*}} \overline{d}(X_k, X)\} \geq \varepsilon \}.$$

Therefore {$r \in N$: $h_r^{-1} \sum_{k\in J_r} \overline{d}(X_k, X) \geq \varepsilon$ }$\in I$.

Hence (X_k)$\in (c_{\theta}^I)_F$.

Theorem 3. *Let ψ be a set of all lacunary sequences.*

(***a***) *If ψ is closed under arbitrary union, then*

$(c_{\mu}^I)_F = \bigcap_{\theta\in\psi} (c_{\theta}^I)_F$, where $\mu = \bigcup_{\theta\in\psi} \theta$;

(***b***) *If ψ is closed under arbitrary intersection, then* $(c_{\nu}^I)_F = \bigcup_{\theta\in\psi} (c_{\theta}^I)_F$, where $\upsilon = \bigcap_{\theta\in\psi} \theta$;

(***c***) *If ψ is closed under union and intersection , then* $(c_{\mu}^I)_F \subseteq (c_{\theta}^I)_F \subseteq (c_{\nu}^I)_F$.

Proof. (***a***) By hypothesis we have $\mu\in\psi$ which is a refinement of each $\theta \in \psi$. Then from theorem 2, we have if (X_k)$\in (c_{\mu}^I)_F$ implies that (X_k)$\in (c_{\theta}^I)_F$.

Thus for each $\theta\in\psi$, we get $(c_{\mu}^I)_F \subseteq (c_{\theta}^I)_F$. The reverse inclusion is obvious.

Hence $(c_{\mu}^I)_F = \bigcap_{\theta\in\psi} (c_{\theta}^I)_F$.

(***b***) By part (***a***) and theorem 2, we have: $(c_{\nu}^I)_F = \bigcup_{\theta\in\psi} (c_{\theta}^I)_F$.

(***c***) By part (***a***) and (***b***), we get: $(c_{\mu}^I)_F \subseteq (c_{\theta}^I)_F \subseteq (c_{\nu}^I)_F$.

REFERENCES

1. Demirci, K. 2002. "On Lacunary Statistical Limit Points". *Demonstrato Mathematica*. 35(1):93-101.

2. Freedman, A.R., Sember, J.J., and Raphael, M.. 1978. "Some Cesaro-type Summability Spaces". *Proc. Lond. Math. Soc.*; 37:508-520.

3. Kostyrko, M., Šalàt, T., and Wilczyński, W. 2000-2001. "I – Convergence". *Real Analysis Exchange*. 26(2):669-686.

4. Nanda, S. 1989. "On Sequences of Fuzzy Numbers". *Fuzzy Sets and Systems*. 33:123-126.
5. Nuray, S. and Savas, E. 1995. "Statistical Convergence of Sequences of Fuzzy Real Numbers". *Math. Slovaca*. 45(3):269-273.

6. Šalàt, T., Tripathy, B.C., and Ziman, M. 2004. "On Some Properties of I-convergence". *Tatra Mt. Math. Publ*. 28: 279-286.

7. Šalàt, T., Tripathy, B.C., and Ziman, M. 2005. "*On I-Convergence Field*". *Indian Jour. Pure Applied Math*. 17: 45-54.

8. Zadeh, L.A. 1965. "Fuzzy Sets". *Information and Control*. 8:338-353.

SUGGESTED CITATION

Hzaarika, B. 2009. "Lacunary *I*-Convergent Sequences of Fuzzy Real Numbers". *Pacific Journal of Science and Technology*. 10(2):203-206.

Pacific Journal of Science and Technology