Some New Type of Lacunary Difference Sequence Spaces Defined by Sequence of Orlicz Functions.

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ABSTRACT

In this article we introduce the sequence spaces $[N_\xi$, **M**, Δ^m, p, q, $v]_1$, $[N_\xi$, **M**, Δ^m, p, q, $v]_0$, and $[N_\xi$, **M**, Δ^m, p, q, $v]_\infty$ and study their algebraic and topological properties. Also we obtain some inclusion relations.

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INTRODUCTION

Throughout w, c, c_0, ℓ_∞ denote the spaces of all, convergent, null, and bounded sequences, respectively. The zero sequence is denoted by θ.

A *lacunary sequence* $\xi = (k_r)$, r = 1, 2, 3, - - -, where k_0=0, is an increasing sequence of non negative integers with $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. We denote $I_r = (k_{r-1}, k_r]$ and $\phi_r = \dfrac{k_r}{k_{r-1}}$, for r = 1, 2, 3, - - -.

The space of lacunary strongly convergent sequences N_ξ was defined by Freedman, Sember and Raphael [5] as follows:

$N_\xi = \{(x_k): \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} |x_k - L| = 0$, for some $L\}$.

The space N_ξ is a BK- space with norm

$$\|(x_k)\|_\xi = \sup_r h_r^{-1} \sum_{k\in I_r} |x_k|$$

N_ξ^0 denotes the subset of those sequences in N_ξ for which $L = 0$.

$(N_\xi^0, \|.\|_\xi)$ is a BK-space.

There is a relation between N_ξ and $|\sigma_1|$of strongly Cesàro summable sequences (see Freedman *et. al* [5]). The space $|\sigma_1|$ is defined as:

$|\sigma_1| = \{(x_k): \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0$, for some $L\}$.

For $\xi = (2^r)$, $|\sigma_1| = N_\xi$.

The idea of difference sequence spaces was introduced by Kizmaz [7] and this subject was generalized by Et and Colak [3] as follows:

$Z(\Delta^m) = \{(x_k) \in w: \Delta^m x_k \in Z\}$,

for $Z = c$, c_0, ℓ_∞, where $m \in N$; $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$; $\sum_{\nu=0}^{m} (-1)^\nu x_{k+\nu}$ and $\Delta^0 x_k = x_k$, for all $k \in N$.

An Orlicz function is a function M:[0,∞)→[0, ∞), which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$, for $x>0$ and $M(x)\to\infty$, as $x\to\infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of x, if there exists a constant $K>0$, such that $M(2x)\le KM(x)$, for all $x\ge 0$. The Δ_2-condition is equivalent to $M(Dx)\le KDM(x)$, for all $x>0$ and for $D>1$.

Lindenstrauss and Tzafriri [8] studied some Orlicz type sequence spaces. They proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p ($1\le p<\infty$). Later on, different classes of sequence spaces defined by Orlicz function were studied by a number of workers on sequence spaces.

DEFINITIONS AND NOTATIONS

The main aim of this article is to introduce the following sequence spaces and examine some properties of the resulting sequence spaces.

Let $p=(p_k)$ denote the sequences of positive real numbers, for all $k\in N$. Let **M** = (M_k) be a sequence of Orlicz functions and $v=(v_k)$ be any sequence of non-zero complex numbers v_k. Let X be a semi-normed space over the field of complex numbers with the semi norm q. $w(X)$ denotes the space of all sequences $x=(x_k)$, where $x_k\in X$. Then we define the following sequence spaces:

$$[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1 = \{(x_k)\in w(X): \lim_{r\to\infty} h_r^{-1}\sum_{k\in I_r}\left[M_k\left(q\left(\frac{v_k\Delta^m x_k - L}{\rho}\right)\right)\right]^{p_k}\to 0, \text{ for some } \rho>0 \text{ and } L\in C\};$$

$$[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0 = \{(x_k)\in w(X): \lim_{r\to\infty} h_r^{-1}\sum_{k\in I_r}\left[M_k\left(q\left(\frac{v_k\Delta^m x_k}{\rho}\right)\right)\right]^{p_k}\to 0, \text{ for some } \rho>0\};$$

$$[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty = \{(x_k)\in w(X): \sup_r h_r^{-1}\sum_{k\in I_r}\left[M_k\left(q\left(\frac{v_k\Delta^m x_k}{\rho}\right)\right)\right]^{p_k}<\infty, \text{ for some } \rho>0\}$$

Throughout Z will denote any one of the notation 0, 1, ∞.

DISCUSSION

(*i*) Let $M_k(x)=M(x)$, $p_k=1$, $v_k=1$, for all k in N and $q(x)=|x|$, then the spaces $[N_\xi, M, \Delta^m]_1$, $[N_\xi, M, \Delta^m]_0$ and $[N_\xi, M, \Delta^m]_\infty$ are studied by Tripathy and Mahanta [12].

(*ii*) Let $M_k(x)=M(x)$, $v_k=1$, for all k in N and $m=0$, then the spaces $(W, \xi, M, p, q)_0$, (W, ξ, M, p, q) and $(W, \xi, M, p, q)_\infty$ introduced and studied by Colak, Tripathy and Et [2].

(*iii*) Let $M_k(x)=x$, $p_k=1$, $v_k=1$, for all k in N, $m=0$ and $q(x)=|x|$, then the spaces $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1 = N_\xi$, $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0 = N_\xi^0$ are introduced and studied by Freedman *et. al* [5].

(*iv*) Let $M_k(x)=M(x)$, $v_k=1$, for all k in N, $\xi=(2^r)$, $q(x)=|x|$ and $m=0$, then the spaces $w(M, p)$, $w_0(M, p)$ and $w_\infty(M, p)$ are studied by Parashar and Choudhary [10].

(*v*) Let $M_k(x)=M(x)$ and $v_k=1$, for all k in N, then the spaces $[N_\xi, M, \Delta^m, p, q]_1$, $[N_\xi, M, \Delta^m, p, q]_0$ and $[N_\xi, M, \Delta^m, p, q]_\infty$ studied by Tripathy, Mahanta and Et [14].

(*vi*) Let $M_k(x) = M(x)$, $v_k = 1$, for all k in N and $\xi = (2^r)$, then the spaces $W(\Delta^m, M, p, q)$, $W_0(\Delta^m, M, p, q)$ and $W_\infty(\Delta^m, M, p, q)$ are studied by Tripathy, Et and Altiń [13].

DEFINITION 1. A sequence space E is said to be *solid* (or *normal*) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \le 1$, for all $k \in N$.

DEFINITION 2. A sequence space E is said to be *symmetric* if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where π is a permutation of N.

DEFINITION 3. A sequence space E is said to be *convergence free* if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Let $K = \{k_1 < k_2 < - - -\} \subseteq N$ and E be a sequence space. A *K-step space* of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}$.

A canonical preimage of a sequence $\{x_{k_n}\} \in \lambda_K^E$ is a sequence $\{y_n\} \in w$ defined as:

$$y_n = \begin{cases} x_n & \text{if } n \in K \\ 0 & \text{otherwise} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, *i.e.* y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

DEFINITION 4. A sequence space E is said to be *monotone* if it contains the canonical preimages of its step spaces.

The following results will be used for establishing some results of this article.

LEMMA 1 [KAMTHAN and GUPTA [6](p 53)]. *A sequence space E is solid implies E is monotone*.

LEMMA 2 [FREEDMAN, SEMBER and RAPHAEL[5], LEMMA 2.1]. *In order for* $|\sigma_1| \subseteq N_\xi$ *it is necessary and sufficient that* $\liminf_r \phi_r > 1$.

LEMMA 3 [FREEDMAN, SEMBER and RAPHAEL [5], LEMMA 2.2]. *In order for* $N_\xi \subseteq |\sigma_1|$ *it is necessary and sufficient that* $\limsup_r \phi_r < \infty$.

LEMMA 4 [ET and NURAY [4], THEOREM 2.2]. *If X is a Banach space normed by* ||.||, *then* $\Delta^m(X)$ *is also a Banach space normed by* :

$$\| x \|_\Delta = \sum_{i=1}^{m} | x_i | + f(\Delta^m x).$$

MAIN RESULTS

THEOREM 1. *Let the sequence* $(p_k) \in \ell_\infty$. *Then* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_1$, $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ *and* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_\infty$ *are linear spaces*.

Proof. The proof is easy, so omitted.

THEOREM 2. *Let* **M** = (M_k) *be a sequence of Orlicz functions and* $p = (p_k)$ *in* ℓ_∞*,of strictly positive real numbers and* $\xi = (k_r)$ *be a lacunary sequence, then* $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ *is a paranormed space*(*not totally paranormed*) *with*

$$g_\Delta(x) = \sum_{i=1}^{m} | x_i | + \inf \{ \rho^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m x_k}{\rho} \right) \right) \right] \le 1, \text{ for some } \rho > 0$$

and $r = 1, 2, 3 - - -\}$,

where $H = \max(1, \sup p_k)$.

PROOF. Clearly $g_\Delta(x) = g_\Delta(-x)$. Since $M_k(0) = 0$, for all $k \in N$, we get $g_\Delta(\overline{\theta}) = 0$, for $x = \overline{\theta}$.

Let $x = (x_k)$ and $y = (y_k)$ be two elements in $[N_\xi, \mathbf{M}, \Delta^m, p, q, v]_0$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m x_k}{\rho_1} \right) \right) \right] \le 1, \; r = 1, 2, 3, ---$$

and $$\sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m y_k}{\rho_2} \right) \right) \right] \le 1, \; r = 1, 2, 3, ---$$

Let $\rho = \rho_1 + \rho_2$, then we have

$$\sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m (x_k + y_k)}{\rho} \right) \right) \right]$$

$$\le \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m x_k}{\rho_1} \right) \right) \right] + \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m y_k}{\rho_2} \right) \right) \right] \le 1$$

Since $\rho > 0$, so we have

$$g_\Delta(x + y) = \sum_{i=1}^m | x_i + y_i | + \inf\{ \rho^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m (x_k + y_k)}{\rho} \right) \right) \right] \le 1, \text{ for some } \rho > 0 \text{ and } r = 1, 2, 3, ---\}$$

$$\le \sum_{i=1}^m | x_i | + \inf\{ \rho_1^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m x_k}{\rho_2} \right) \right) \right] \le 1, \text{ for some } \rho_1 > 0 \text{ and } r = 1, 2, 3 ---\}$$

$$+ \sum_{i=1}^m | y_i | + \inf\{ \rho_2^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m y_k}{\rho_2} \right) \right) \right] \le 1, \text{ for some } \rho_2 > 0 \text{ and } r = 1, 2, 3, ---\}$$

$$= g_\Delta(x) + g_\Delta(y)$$

ie $g_\Delta(x + y) \le g_\Delta(x) + g_\Delta(y)$.

Finally, let $\lambda \in C$ be a given non zero scalar in C. The continuity of the product follows from the following expression.

$$g_\Delta(\lambda x) = \sum_{i=1}^m | \lambda x_i | + \inf\{ \rho^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m \lambda x_k}{\rho} \right) \right) \right] \le 1, \text{ for some } \rho > 0 \text{ and } r = 1, 2, 3, ---\}$$

$$= \lambda \sum_{i=1}^m | x_i | + \inf\{ (| \lambda | \eta)^{\frac{p_k}{H}} : \sup_r h_r^{-1} \sum_{k \in I_r} \left[M_k \left(q \left(\frac{v_k \Delta^m x_k}{\eta} \right) \right) \right] \le 1, \text{ for some } \rho > 0 \text{ and } r = 1, 2, 3, ---\}$$

where $\eta = \dfrac{\rho}{|\lambda|} > 0$. This completes the proof.

The proof of the following result is easy, so omitted.

THEOREM 3. *Let* **M** = (M_k) *and* **S** = (S_k) *be sequences of Orlicz functions*. *For* $p = (p_k) \in \ell_\infty$, *of strictly positive real numbers*. *Then* (*i*) $[N_\xi$, **M**, Δ^m, $p, q, v]_Z \subseteq [N_\xi$, **S.M**, Δ^m, $p, q, v]_Z$;

(*ii*) $[N_\xi$, **M**, Δ^m, $p, q, v]_Z \cap [N_\xi$, **S**, Δ^m $p, q, v]_Z \subseteq [N_\xi$, **S + M**, Δ^m, $p, q, v]_Z$.

THEOREM 4. *The inclusions* $[N_\xi$, **M**, Δ^{m-1}, $q]_Z \subseteq [N_\xi$, **M**, Δ^m, $q]_Z$, *for* $m \geq 1$. *In general* $[N_\xi$, **M**, Δ^i, $q]_Z \subseteq [N_\xi$, **M**, Δ^m, $q]_Z$, *for* i = 0, 1, 2, 3, - - -m-1 *and the inclusions are strict*.

PROOF. Let $(x_k) \in [N_\xi$, **M**, Δ^{m-1}, $q]_0$. Then

$$\lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \left[M_k\left(q\left(\frac{\Delta^{m-1}x_k}{\rho}\right)\right)\right]=0, \text{ for some } \rho>0.$$

Since **M** is non decreasing and convex, so we have

$$h_r^{-1} \sum_{k\in I_r} \left[M_k\left(q\left(\frac{\Delta^{m}x_k}{2\rho}\right)\right)\right]=$$

$$h_r^{-1} \sum_{k\in I_r} \left[M_k\left(q\left(\frac{\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1}}{2\rho}\right)\right)\right]$$

$$\leq h_r^{-1}\left\{\sum_{k\in I_r} \left[M_k\left(q\left(\frac{\Delta^{m-1}x_k}{2\rho}\right)\right)\right]+\sum_{k\in I_r} \left[M_k\left(q\left(\frac{\Delta^{m-1}x_{k+1}}{2\rho}\right)\right)\right]\right\}$$

$$\leq$$

$$h_r^{-1} \sum_{k\in I_r} \tfrac{1}{2}\left[M_k\left(q\left(\frac{\Delta^{m-1}x_k}{\rho}\right)\right)\right]+h_r^{-1} \sum_{k\in I_r} \tfrac{1}{2}\left[M_k\left(q\left(\frac{\Delta^{m-1}x_{k+1}}{\rho}\right)\right)\right]$$

$$<$$

$$h_r^{-1} \sum_{k\in I_r} \left[M_k\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]+h_r^{-1} \sum_{k\in I_r} \left[M_k\left(q\left(\frac{\Delta^{m}x_k}{\rho}\right)\right)\right]$$

Taking limit as $r\to\infty$, we have

$$\lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \left[M_k\left(q\left(\frac{\Delta^{m-1}x_k}{\rho}\right)\right)\right]= 0$$

i.e. $(x_k) \in [N_\xi$, **M**, Δ^m, $q]_0$.

The rest of the cases follow in similar way.
By using induction, we have:

$[N_\xi$, **M**, Δ^i, $q]_Z \subseteq [N_\xi$, **M**, Δ^m, $q]_Z$, for i = 0, 1, 2, 3, - - - m-1.

The above inclusion is strict follows from the following example.

EXAMPLE 1. Let $M_k(x) = x^2$, for all $x\in[0,\infty)$, $q(x) = |x|$ and $\xi = (2^r)$, for all k. Consider a sequence (x_k) defined as:

$(x_k) = (k^{m-1}, k^{m-1}, k^{m-1}, - - -)$.

Then $\Delta^m x_k = 0$, but $\Delta^{m-1}x_k = (-1)^{m-1}(m-1)!$, for all $k\in N$.

Thus $(x_k)\in[N_\xi$, **M**, Δ^m, $q]_0$, but $(x_k)\notin[N_\xi$, **M**, Δ^{m-1}, $q]_0$.

THEOREM 5. *Let* $p=(p_k)$, $q=(q_k)$ *in* ℓ_∞, *be two sequences of strictly positive real numbers with* $0 < p_k \leq t_k$, *for all* $k\in N$ *and* $\left(\frac{p_k}{t_k}\right) \in \ell_\infty$. *Then* $[N_\xi$, **M**, Δ^m, $t, q, v]_Z \subseteq [N_\xi$, **M**, Δ^m, $p, q, v]_Z$.

PROOF. If we take $z_k = \sum_{k\in I_r}\left[M_k\left(q\left(\frac{v_k\Delta^m x_k - L}{\rho}\right)\right)\right]^{q_k}$, for all $r\in N$, then the proof of this result follows from the same technique of Theorem 5 [8].

THEOREM 6. *Let* $\xi = (k_r)$ *be a lacunary sequence*.

(*i*) *Let* **M** = (M_k) *be a sequence of Orlicz functions*. *Then* $[N_\xi$, **M**, Δ^m, $p, q, v]_0 \subset [N_\xi$, **M**, Δ^m, $p, q, v]_1 \subset [N_\xi$, **M**, Δ^m, $p, q, v]_\infty$, *and the inclusion is strict*.

(*ii*) *If* $|v_k| \leq 1$, *then* $[N_\xi$, **M**, Δ^m, $p, q, v]_Z \subseteq [N_\xi$, **M**, Δ^m, $p, q]_Z$, *for* $Z = 0, \infty$.

PROOF. (*i*) The inclusion $[N_\xi$, **M**, Δ^m, *p*, *q*, $v]_0 \subset [N_\xi$, **M**, Δ^m, *p*, *q*, $v]_1$ is obvious. Let (x_k) be a sequence of $[N_\xi$, **M**, Δ^m, *p*, *q*, $v]_1$. Then there exists ρ >0 such that:

$$h_r^{-1}\sum_{k\in I_r}\left[M_k\left(q\left(\frac{v_k\Delta^m x_k - L}{\rho}\right)\right)\right]^{p_k} \to 0, \text{ as } r\to\infty.$$

Since M_k is non- decreasing and convex, so we have

$$h_r^{-1}\sum_{k\in I_r}\left[M_k\left(q\left(\frac{v_k\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} \le D$$

$$h_r^{-1}\sum_{k\in I_r}\left[M_k\left(q\left(\frac{v_k\Delta^m x_k - L}{\rho}\right)\right)\right]^{p_k} + D$$

$$\max\left[1, M_k\left(q\left(\tfrac{L}{\rho}\right)\right)\right]^H,$$

where $G = \sup_k p_k$, $D = \max(1, 2^{G-1})$.

Then (x_k) belong to $[N_\xi$, **M**, Δ^m, *p*, *q*, $v]_\infty$.

The inclusions are strict follows from the following example.

EXAMPLE 2. Let p_k = 4, for *k* even and p_k =5, for *k* odd. Let $m \ge 0$ be given. Let $v_k = k$, $M_k(x) = x^2$ for all $k \in N$ and $q(x) = |x|$. Let $\xi = (2^r)$ be a lacunary sequence.

Consider a sequence (x_k) defined as:
$(x_k) = (k^m, k^m, k^m, k^m, ---)$.

Then (x_k) belong to $[N_\xi$, **M**, Δ^m, *p*, *q*, $v]_1$ but does not belong to $[N_\xi$, **M**, Δ^m, *p*, *q*, $v]_0$.

The proof of (*ii*) is easy, so omitted.

THEOREM 7. *Let* **M** = (M_k) *and* **S** = (S_k) *be sequences of Orlicz functions. If* M_k *and* S_k *are equivalent for each* $k \in N$ *and* $\xi = (k_r)$ *be a lacunary sequence. Then:*

$[N_\xi$, **M**, Δ^m, *p*, *q*, $v]_Z$ = $[N_\xi$, **S**, Δ^m, *p*, *q*, $v]_Z$.

The proof of this result easy, so omitted.

THEOREM 8. *Let* **M** = (M_k) be a sequence of Orlicz functions and let q_1 and q_2 be semi norms. Then:

(*i*) $[N_\xi$, **M**, Δ^m, *p*, q_1, $v]_Z \cap [N_\xi$, **M**, Δ^m *p*, q_2, $v]_Z \subseteq$ $[N_\xi$, **M**, Δ^m, *p*, q_1+q_2, $v]_Z$

(*ii*) $[N_\xi$, **M**, Δ^m, *p*, q_1, $v]_Z \subseteq$ $[N_\xi$, **M**, Δ^m, *p*, q_2, $v]_Z$.

The proof of this result easy, so omitted.

PROPOSITION 9. *Let* $\xi = (k_r)$ *be a lacunary sequence*.

(*i*) *If* $\liminf_r \phi_r > 1$, *then for any sequence of Orlicz functions* **M** = (M_k), *for all* $k \in N$, [*w*, **M**, Δ^m, *p*, *q*, $v]_0 \subseteq [N_\xi$, **M**, Δ^m, *p*, *q*, $v]_0$, *where* [*w*, **M**, Δ^m, *p*, *q*, $v]_0$ =

$$\left\{(x_k) \in w(X) : \lim_{n\to\infty}\sum_{k=1}^{n}\left[M_k\left(q\left(\frac{v_k\Delta^m x_k}{\rho}\right)\right)\right]^{p_k} = 0, \text{ for some } \rho > 0\right\}$$

(*ii*) *If* $\limsup_r \phi_r < \infty$, *then for any sequence of Orlicz functions* **M** = (M_k), *for all* $k \in N$, $[N_\xi$, **M**, Δ^m, *p*, *q*, $v]_0 \subseteq$ [*w*, **M**, Δ^m, *p*, *q*, $v]_0$.

PROPOSITION 10. *Let* $\xi = (k_r)$ *be a lacunary sequence, with* $0 < \liminf_r \phi_r \le \limsup_r \phi_r < \infty$, *then for any sequence of Orlicz functions* **M** = (M_k), *for all* $k \in N$,

$[N_\xi$, **M**, Δ^m, *p*, *q*, $v]_0$ = [*w*, **M**, Δ^m, *p*, *q*, $v]_0$.

RESULT 11. *The spaces* $[N_\xi$, **M**, p, q, $v]_0$ *and* $[N_\xi$, **M**, p, q, $v]_\infty$ *are solid as well as monotone*. *The spaces* $[N_\xi$, **M**, Δ^m, p, q, $v]_Z$ *are not solid in general*.

PROOF. Let $(x_k) \in [N_\xi$, **M**, p, q, $v]_0$. Then there exists $\rho > 0$ such that:

$$\lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \left[M_k\left(q\left(\frac{v_k x_k}{\rho}\right)\right)\right]^{p_k} = 0.$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \le 1$, for all $k \in N$.

Since $|\alpha_k|^{p_k} \le \max(1, |\alpha_k|^G) \le 1$, for all $k \in N$, where $G = \sup_k p_k < \infty$.

Then for each r, we have:

$$h_r^{-1} \sum_{k\in I_r} \left[M_k\left(q\left(\frac{\alpha_k (v_k x_k)}{\rho}\right)\right)\right]^{p_k} \le$$

$$h_r^{-1} \sum_{k\in I_r} \left[M_k\left(q\left(\frac{v_k x_k}{\rho}\right)\right)\right]^{p_k} \qquad \textbf{(1)}$$

Therefore $(\alpha_k x_k) \in [N_\xi$, **M**, p, q, $v]_0$.

Hence $[N_\xi$, **M**, p, q, $v]_0$ is solid. From Lemma 1, we have $[N_\xi$, **M**, p, q, $v]_0$ is monotone.

Again from inequality (1) and Lemma 1, we have $[N_\xi$, **M**, p, q, $v]_\infty$ is solid as well as monotone.

In order to prove that the spaces $[N_\xi$, **M**, Δ^m, p, q, $v]_1$ and $[N_\xi$, **M**, Δ^m, p, q, $v]_\infty$ are not solid in general, we consider the following example.

EXAMPLE 3. Let $M_k(x) = x^t$, for all $k \in N$ and $t \ge 1$. Let $p_k = \frac{1}{k}$ and $v_k = k$, for all $k \in N$ and $q(x) = |x|$. Let the lacunary sequence $\xi = (2^r)$, for all $r \in N$.

Consider a sequence (x_k) as $x_k = k^2$, for all $k \in N$. Then (x_k) belongs to $[N_\xi$, **M**, Δ^m, p, q, $v]_1$ and $[N_\xi$, **M**, Δ^m, p, q, $v]_\infty$, for $m = 1$.

Let $(\alpha_k) = ((-1)^k)$, for all $k \in N$. Then $(\alpha_k x_k)$ does not belongs to $[N_\xi$, **M**, Δ^m, p, q, $v]_1$ and $[N_\xi$, **M**, Δ^m, p, q, $v]_\infty$.

Next to show that the space $[N_\xi$, **M**, Δ^m, p, q, $v]_0$ is not solid in general. We consider the following example.

EXAMPLE 4. Under the restrictions on **M**, p, v, m, q and ξ as in Example 3.2. Consider the sequence (x_k) as $x_k = 2$, for all $k \in N$. Let $(\alpha_k) = ((-1)^k)$, for all $k \in N$. Then $(\alpha_k x_k)$ does not belongs to $[N_\xi$, **M**, Δ^m, p, $v]_0$.

Hence, the space $[N_\xi$, **M**, Δ^m, p, $v]_0$ is not solid.

RESULT 12. *The space* $[N_\xi$, **M**, p, q, $v]_1$ *is not monotone as such is not solid*.

PROOF. The space $[N_\xi$, **M**, p, q, $v]_1$ is not monotone follows from the following example.

EXAMPLE 5. Let $p_k = 1 + \frac{1}{k^2}$ and $v_k = k$, for all $k \in N$. Let $M_k(x) = x^t$, for all $k \in N$ and $t \ge 1$ and $q(x) = |x|$. Let lacunary sequence $\xi = (2^r)$, for all $r \in N$.

Consider a sequence $(x_k) = (2, 2, 2, 2, - - -)$, for all $k \in N$.

Consider the K^{th}- step space E_k for a sequence space E defined as for $(x_k) \in E$, (y_k) is the K^{th} canonical preimages of (x_k) *i e*

$(y_k) \in E_k$ implies $y_k = x_k$, if k is even and $y_k = 0$, otherwise. Then y_k does not belong to E. Hence the space $[N_\xi$, **M**, $p, q, v]_1$ is not monotone.

RESULT 13. *The spaces* $[N_\xi$, **M**, $\Delta^m, p, q, v]_Z$ *are not monotone in general*.

PROOF. The result follows from the examples 3.3 and 3.4, on considering the K^{th}- step space E_k for a sequence space E defined as for $(x_k) \in E$, then $(y_k) \in E_k$ implies $y_k = x_k$, if k is even and $y_k = 0$, otherwise. Then the sequence (x_k) of the example 3.3 belongs to $[N_\xi$, **M**, Δ^m, $p, q, v]_Z$, for $Z = 1, \infty$ and (y_k) does not belong to $[N_\xi$, **M**, $\Delta^m, p, q, v]_Z$, for $Z = 1, \infty$.

Similarly the sequence (x_k) of the example 3.4 belongs to $[N_\xi$, **M**, $\Delta^m, p, q, v]_0$ but (y_k) does not belong to $[N_\xi$, **M**, $\Delta^m, p, q, v]_0$.

RESULT 14. *The spaces* $[N_\xi$, **M**, $\Delta^m, p, q, v]_Z$ *are not symmetric in general*.

PROOF. The spaces are not symmetric in general follows from the following example.

EXAMPLE 6. Let $M_k(x) = x^2$, $p_k = k$ and $v_k = k^2$, for all $k \in N$ and $q(x) = |x|$. Let the lacunary sequence $\xi = (2^r)$, for all $r \in N$.

Consider a sequence (x_k), defined by $x_k = k^3$, for all $k \in N$.

Then (x_k) belong to $[N_\xi$, **M**, $\Delta^m, p, q, v]_0$, for $m = 1$.

Consider the sequence (y_k) which is a rearrangement of the sequence (x_k) defined as:

$(y_k) = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, ---)$.

Then (y_k) does not belong to $[N_\xi$, **M**, $\Delta^m, p, q, v]_Z$.

Hence the spaces $[N_\xi$, **M**, $\Delta^m, p, q, v]_Z$ are not symmetric in general.

RESULT 15. *The space* $[N_\xi$, **M**, $\Delta^m, p, q, v]_0$ *is not convergence free*.

Proof of the result follows from the following example.

EXAMPLE 7. Let $M_k(x) = x$, $p_k = k$ and $v_k = k$, for all $k \in N$ and $q(x) = |x|$. Let the lacunary sequence $\xi = (2^r)$, for all $r \in N$.

Consider a sequence (x_k), defined as:

$$x_k = \begin{cases} 2, \text{ if } \ k \text{ even}; \\ 0, \text{ if } \ k \text{ odd} \end{cases}$$

Then (x_k) belong to $[N_\xi$, **M**, $\Delta^m, p, q, v]_0$, for $m = 2$.

Consider the sequence (y_k) defined as:

$$y_k = \begin{cases} k^2, \text{if } \ k \text{ even}; \\ 0, \text{ if } \ k \text{ odd}. \end{cases}$$

Then (y_k) does not belong to $[N_\xi$, **M**, $\Delta^m, p, q, v]_0$.

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