Boundedness and Stability of Solutions of Some Nonlinear Differential Equations of the Third-Order.

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ABSTRACT

Sufficient conditions are established for the uniform ultimate boundedness of solutions of a third-order nonlinear differential equation (1). When $p(t,x,x',x'')=0$, criteria under which all solutions $x(t)$, its first and second derivatives tend to zero as $t\to\infty$, are given.

(Keywords: third-order, differential equations, stability, uniform-bounded, ultimate boundedness)

INTRODUCTION

Nonlinear third-order differential equations have been extensively studied with high degree of generality. In particular, there have been interesting works on asymptotic behavior, boundedness, periodicity, and stability of solutions for nonlinear differential equations of the third-order. Authors that have worked in this direction include Ademola *et. al.,* [1, 2, 3], Afuwape [4], Bereketoğlu and Giyöri [5], Ezeilo [6], Omeike [7], and Swick [9], to mention a few.

All the above mention works were done by using the Lyapunov's second method except in [2] and [4], where Yoshizawa function and frequency domain technique were used.

In this paper, we shall investigate uniform ultimate boundedness and stability of solutions of the third-order nonlinear ordinary differential equation:

$$x''' + f(t,x,x',x'')x'' + q(t)g(x,x') + h(x,x',x'') = p(t,x,y,z) \quad \textbf{(1)}$$

or its equivalent system:

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= p(t,x,y,z) - f(t,x,x',x'')x'' - q(t)g(x,x') \\ &\quad - h(x,x',x'') \end{aligned} \quad \textbf{(2)}$$

where f, g, h, p and q are continuous in their respective arguments, and x', x'' and x''' denote the first, second and third derivatives of the function $x(t)$ with respect to t. The derivatives:

$$\begin{aligned} &\partial f(t,x,y,z)/\partial t = f_t(t,x,y,z), \\ &\partial f(t,x,y,z)/\partial x = f_x(t,x,y,z), \ \partial f(t,x,y,z)/\partial z = f_z(t,x,y,z), \\ &\partial g(x,y)/\partial x = g_x(x,y), \ \partial h(x,y,z)/\partial x = h_x(x,y,z), \\ &\partial h(x,y,z)/\partial y = h_y(x,y,z), \ \partial h(x,y,z)/\partial z = h_z(x,y,z) \end{aligned}$$

and $dq(t)/dt = q'(t)$

exist and are continuous. Moreover, the existence and uniqueness of solutions of (1) will be assumed.

In 2005, Tunç [10] discussed criteria for boundedness of solutions of a third-order nonlinear differential equation:

$$x''' + f(x,x',x'')x'' + g(x,x') + h(x,x',x'') = p(t,x,x',x'') \quad \textbf{(3)}$$

In 2008, Ademola *et. al.,* [2] and Omeike [7] established conditions for the ultimate boundedness of solutions of a third-order differential equation (3) using a complete Yoshizawa and a complete Lyapunov functions, respectively.

However, the problem of stability and boundedness of solutions of third-order differential equations where the nonlinear, specifically the restoring, terms depend on the independent variable t and multiple of the functions of t are scare. Motivation for this study comes from the works of Ademola *et. al,* [1, 3], Omeike [7] and Swick [9].

The purpose of this paper, therefore, is to investigate criteria under which all solutions $x(t)$, its first and second derivative, when $p(t,x,x',x'')=0$, tend to zero as $t\to\infty$.

Sufficient conditions were also obtained for uniform ultimate boundedness of solutions of a third-order differential Equation (1). Here, the Lyapunov second method is used to achieve the desired results. Our results do not only generalize, to third-order equation, the results in [1, 3, 9] but also include and extend the result in [7]. Some existing results on third-order nonlinear differential equations, which have been discussed in [8], are also generalized.

MAIN RESULTS

In the case $p(t,x,x',x'')=0$, Equation (2) becomes:

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= -f(t,x,x',x'')x'' - q(t)g(x,x') - h(x,x',x'') \end{aligned} \quad \textbf{(4)}$$

with the following result.

THEOREM 1. In addition to the basic assumptions on the functions f, g, h, p and q, suppose that there are positive constants $a, a_0, b, b_0, c, q_0, \alpha, \beta, \delta$ and μ such that for all $t\ge 0$, the following conditions are satisfied:

(i) $h(0,0,0)=0,\ \delta\le h(x,0,0)/x \quad x\ne 0;$

(ii) $g(0,0)=0,\ b\le g(x,y)/y\le b_0$ for all $x, y\ne 0;$

(iii) $a\le f(t,x,y,z)\le a_0$ for all $x,y,z;$

(iv) $\mu\le q(t),\ \ q'(t)\le 0;$

(v) $f_t(t,x,y,z)\le 0,\ \ yf_x(t,x,y,z)\le 0,$ $g_x(x,y)\le 0,\ h_x(x,0,0)\le c$ for all $x,y\ne 0;$

(vi) $h_y(x,y,0)\ge 0,\ h_z(x,0,z)\ge 0,$ $yf_z(t,x,y,z)\ge 0$ for all x,y,z.

Then every solution $(x(t),y(t),z(t))$ of (4) is uniform-bounded and satisfies:

$$x(t)\to 0,\quad y(t)\to 0,\quad z(t)\to 0 \ \text{ as } t\to\infty. \quad \textbf{(5)}$$

REMARK 2. Observe that the hypotheses: $a\le f(t,x,y,z)$, $b\le g(x,y)/y \quad y\ne 0$, $\delta\le h(x,0,0)/x \quad x\ne 0$, $h_x(x,0,0)\le c$ and $\mu\le q(t)$ of Theorem 1 imply the existence of arbitrary positive constants α and β satisfying:

$$\frac{c}{b\mu}<\alpha<a \quad \textbf{(6a)}$$

and

$$0\le\beta\le\min\{b\mu;(ab\mu-c)\eta_1;\frac{1}{2}(a-\alpha)\eta_2\} \quad \textbf{(6b)}$$

where,

$$\eta_1=[1+a+\delta^{-1}\mu^2[g(x,y)/y-b]^2]^{-1}$$

and $\eta_2=[1+\delta^{-1}[f(t,x,y,z)-a]^2]^{-1}$ for all x,y,z and $t\ge 0$.

REMARK 3.

(i) Note that $f(t,x,y,z)\equiv f(z),\ q(t)g(x,y)\equiv g(y)$ and $h(x,y,x)\equiv h(x)$, system (2.1) reduces to that investigated by Ademola *et. al,* in [3].

(ii) Also, whenever $f(t,x,y,z)\equiv f(t,x,y),\ g(x,y)\equiv g(y)$ and $h(x,y,z)\equiv r(t)h(x)$ system (2.1) specializes to that studied by Swick in [9].

(iii) Furthermore, the hypotheses on (4) are considerably weaker than those in [3] and [9].

Hence, our result generalizes the results in [3] and [9].

The proofs of our results depend on some certain fundamental properties of a continuously differentiable Lyapunov function $V = V(t, x, y, z)$ defined by:

$$2V = 2(\alpha + a)\int_0^x h(\xi, 0, 0)d\xi + 4q(t)\int_0^y g(x, \tau)d\tau$$
$$+4h(x, 0, 0)y + 2z^2 + 2\beta xz$$
$$+2(\alpha + a)\int_0^y \tau f(t, x, \tau, 0)d\tau + 2(\alpha + a)yz$$
$$+\beta y^2 + b\beta q(t)x^2 + 2a\beta xy$$

(7)

where α and β are defined in (6). Namely, this function and its time derivative satisfy some fundamental inequalities which are discussed in the following lemmas.

LEMMA 4. Subject to the hypotheses of Theorem 1, $V(t,0,0,0) = 0$ and there are positive constants $D_0 = D_0(a,b,c,\alpha,\beta,\delta,\mu)$ and $D_1 = D_1(a,b,c,a_0,b_0,q_0,\alpha,\beta,\delta)$ such that:

(i) $D_0(x^2 + y^2 + z^2) \le V(t, x(t), y(t), z(t)) \le D_1(x^2 + y^2 + z^2);$

(ii) $V(t, x(t), y(t), z(t)) \to \infty$ as $x^2 + y^2 + z^2 \to \infty.$

Furthermore, for any solution $(x(t), y(t), z(t))$ of (4)

(iii) $V' \equiv \frac{d}{dt}V(t, x(t), y(t), z(t)) \le -D_0(x^2 + y^2 + z^2).$

PROOF. It is clear that $V(t,0,0,0) = 0$. Since $h(0,0,0) = 0$ and $b \ne 0 \ne q(t)$, we observe that the function V defined in (7) can be rearranged as follows:

$$2V = \frac{2}{bq(t)}\int_0^x [(\alpha + a)bq(t) - 2h_\xi(\xi,0,0)]h(\xi,0,0)d\xi$$
$$+4q(t)\int_0^y [g(x,\tau)/\tau - b]\tau d\tau + \beta y^2 + \beta[bq(t) - \beta]x^2$$
$$+2\int_0^y \tau[(\alpha + a)f(t,x,\tau,0) - (\alpha^2 + a^2)]d\tau + (\alpha y + z)^2$$
$$+(\beta x + ay + z)^2 + \frac{2}{bq(t)}[h(x,0,0) + bq(t)y]^2.$$

(8)

Now, since $\mu \le q(t), h_x(x,0,0) \le c$ and $h(x,0,0)/x \ge \delta \quad x \ne 0,$ if follows that ,

$$\frac{2}{bq(t)}\int_0^x [(\alpha + a)bq(t) - 2h_\xi(\xi,0,0)]h(\xi,0,0)d\xi$$
$$\ge [(\alpha b\mu - c) + (ab\mu - c)]\delta b^{-1}\mu^{-1}x^2.$$

(9a)

Also, $g(x,y)/y \ge b \quad y \ne 0,$ implies that,

$$4q(t)\int_0^y [g(x,\tau)/\tau - b]\tau d\tau \ge 0. \qquad \textbf{(9b)}$$

Furthermore, from the inequalities in condition (iii) of Theorem 1, we obtain:

$$2\int_0^y \tau[(\alpha + a)f(t,x,\tau,0) - (\alpha^2 + a^2)]d\tau$$
$$\ge \alpha(a - \alpha)y^2. \qquad \textbf{(9c)}$$

Combining estimates (9a) - (9c) with (8), we obtain:

$$V \ge \frac{1}{2}\{[(\alpha b\mu - c) + (ab\mu - c)]\delta b^{-1}\mu^{-1} + \frac{1}{2}(\alpha y + z)^2$$
$$+\beta[b\mu - \beta]\}x^2 + \frac{1}{2}[\alpha(a - \alpha) + \beta]y^2$$
$$+\frac{1}{2}(\beta x + ay + z)^2 + b^{-1}\mu^{-1}(\delta x + b\mu y)^2.$$

From estimates (6a) and (6b), we have $\alpha b\mu - c > 0,\ ab\mu - c > 0,\ a-\alpha > 0$ and $b\mu - \beta > 0$. It follows that the V defined in (7) is positive definite. Hence, there exists a positive constant $\delta_0 = \delta_0(a,b,c,\alpha,\beta,\delta,\mu)$ such that:

$$V \geq \delta_0(x^2+y^2+z^2). \quad \textbf{(10)}$$

It is clear from (10) that:

$$V(t,x,y,z) \to \infty \text{ as } x^2+y^2+z^2 \to \infty. \quad \textbf{(11)}$$

Let us observe that $q'(t) \leq 0$ implies, $q(t) \leq q(0) = q_0$ and since $h(0,0,0)=0$ then $h_x(x,0,0) \leq c$ implies $h(x,0,0) \leq cx$ $x \neq 0$. These together with $g(x,y)/y \leq b_0$ $y \neq 0$, $f(t,x,y,z) \leq a_0$ and Schwartz inequality Equation (7) becomes:

$$2|V| \leq (\alpha+a)cx^2 + 2b_0q_0y^2 + 2c(x^2+y^2)$$
$$+(\alpha+a)a_0y^2 + 2z^2 + (\alpha+a)(y^2+z^2)$$
$$+\beta y^2 + b\beta q_0x^2 + a\beta(x^2+y^2) + \beta(x^2+z^2).$$

Rearranging the terms, there exists a positive constant $\delta_1 = \delta_1(a,b,c,a_0,b_0,q_0,\alpha,\beta)$ such that:

$$V \leq \delta_1(x^2+y^2+z^2). \quad \textbf{(12)}$$

To deal with hypothesis (iii) of Lemma 4, let $(x(t),y(t),z(t))$ be any solution of (4) and consider the function $V = V(t,x(t),y(t),z(t))$. By an elementary calculation using (4) and (7), we have:

$$V'_{(2.1)} = W_1 + W_2 - W_3 - W_4 + a\beta y^2 + 2\beta yz$$
$$-\beta q(t)[g(x,y)/y - b]xy - \beta[f(t,x,y,z) - a]xz \quad \textbf{(13)}$$

Where,

$$W_1 = 2q'(t)\left[\int_0^y g(x,\tau)d\tau + \frac{1}{4}b\beta x^2\right],$$

$$W_2 = (\alpha+a)\left[\int_0^y \tau f_t(t,x,\tau,0)d\tau + y\int_0^y \tau f_x(t,x,\tau,0)d\tau\right]$$
$$+2q(t)y\int_0^y g_x(x,\tau)d\tau,$$

$$W_3 = \beta\frac{h(x,y,z)}{x}x^2 + \left[2f(t,x,y,z) + (\alpha+a)\right]z^2$$
$$+\left[(\alpha+a)q(t)\frac{g(x,y)}{y} - 2h_x(x,0,0)\right]y^2,$$

$$W_4 = (\alpha+a)y^2\left[\frac{h(x,y,z)-h(x,0,0)}{y}\right]$$
$$+2z^2\left[\frac{h(x,y,z)-h(x,0,0)}{z}\right]$$
$$+(\alpha+a)yz^2\left[\frac{f(t,x,y,z)-f(t,x,y,0)}{z}\right]$$

By hypothesis (iv) $q'(t) \leq 0$ for all $t \geq 0$. If $q'(t) = 0$ then $W_1 = 0$. For those t's such that $q'(t) < 0$, we have,

$$W_1 = 2q'(t)\left[\int_0^y g(x,\tau)d\tau + \frac{1}{4}b\beta x^2\right] \leq 0$$

since,

$$\int_0^y g(x,\tau)d\tau + \frac{1}{4}b\beta x^2 \geq \frac{1}{2}b\left(\frac{1}{2}\beta x^2 + y^2\right) \geq 0 \text{ f}$$

or all x and y. Thus, on combining the two cases, we have:

$W_1 \leq 0$ for all $t \geq 0$, x and y.

In view of condition (v) of Theorem 1, since a and α are positive constants and $q(t) \geq \mu > 0$, we have $W_2 \leq 0$.

Moreover, $h(x,0,0)/x \geq \delta$ $x \neq 0$, $g(x,y)/y \geq b$ $y \neq 0$, $h_x(x,0,0) \leq c$, $f(t,x,y,z) \geq a$ and $q(t) \geq \mu$, we have

$$W_3 \geq \beta\delta x^2 + \left[(\alpha+a)b\mu - 2c\right]y^2 + \left[a-\alpha\right]z^2.$$

Also, from hypothesis (vi) of Theorem 1, we have the following inequalities:

$$W_{41} = (\alpha + a)y^2 \left[\frac{h(x,y,z) - h(x,0,0)}{y} \right]$$

$$= (\alpha + a)y^2 h_y(x, \theta_1 y, 0) \geq 0,$$

$y \neq 0$ $0 \leq \theta \leq 1$, a and α are positive constants, but $W_{41} = 0$ when $y = 0$. Hence, $W_{41} \geq 0$ for all x and y.

Similarly, when $z \neq 0$, we have :

$$W_{42} = 2z^2 \left[\frac{h(x,y,z) - h(x,0,0)}{z} \right]$$

$$= 2z^2 h_z(x, 0, \theta_2 z) \geq 0,$$

$0 \leq \theta_2 \leq 1$, but $W_{42} = 0$ when $z = 0$.

Hence, $W_{42} \geq 0$ for all x and z.

Finally, when $z \neq 0$, we have:

$$W_{43} = (\alpha + a)yz^2 \left[\frac{f(t,x,y,z) - f(t,x,y,0)}{z} \right]$$

$$= (\alpha + a)z^2 y f_z(x, 0, \theta_2 z) \geq 0$$

$0 \leq \theta_3 \leq 1$, but $W_{43} = 0$ when $z = 0$. Thus $W_{43} \geq 0$ for all x, y, z and $t \geq 0$.

On combining estimates W_{41}, W_{42} and W_{43}, we obtain $W_4 \geq 0$ for all x, y, z and $t \geq 0$.

On gathering the estimates W_1, W_2, W_3 and W_4 with (13) and complete the squares to get:

$$V'_{(2.1)} \leq -\frac{1}{2}\beta\delta x^2 - (\alpha b \mu - c)y^2 - \frac{1}{2}(a - \alpha)z^2$$

$$-\left[ab\mu - c - \beta \left[1 + a + \delta^{-1}\mu^2 \left[\frac{g(x,y)}{y} - b \right]^2 \right] \right] y^2$$

$$-\left[\frac{1}{2}(a - \alpha) - \beta \left[1 + \delta^{-1}[f(t,x,y,z) - a]^2 \right] \right] z^2$$

$$-\frac{\beta\delta}{4}\left[x + 2\delta^{-1}\mu \left[\frac{g(x,y)}{y} - b \right] y \right]^2$$

$$-\frac{\beta\delta}{4}\left[x + 2\delta^{-1}[f(t,x,y,z) - a]z \right]^2.$$

Since β and δ are positive constants, it follows that $\left[x + 2\delta^{-1}\mu \left[\frac{g(x,y)}{y} - b \right] y \right]^2 \geq 0$ and $\left[x + 2\delta^{-1}[f(t,x,y,z) - a]z \right]^2 \geq 0$ for all x, y, z and $t \geq 0$. Hence, by (6a) and (6b), there exists a positive constant $\delta_2 = \delta_2(a, b, c, \alpha, \beta, \delta, \mu)$ such that:

$$V'_{(2.1)} \leq \delta_2 (x^2 + y^2 + z^2). \qquad \textbf{(14)}$$

This completes the proof of Lemma 4.

PROOF OF THEOREM 1. From hypotheses (i) - (iii) of Lemma 4 it follows that the solution $(x(t), y(t), z(t))$ of (4) is uniform-bounded (see [11] p 38-39). Moreover, from Lemma 4, $V' \leq -D_0(x^2 + y^2 + z^2)$.

Now, let $W(X) \equiv D_0(x^2 + y^2 + z^2)$ a positive definite function with respect to a closed set $\Omega \equiv \{(x,y,z) \mid x = 0,\ y = 0, z = 0\}$ and $V'(t, X) \leq -W(X)$. From the continuity of $h(x,y,z)$ and $q(t)$, and the fact that the functions $f(t,x,y,z)$ and $g(x,y)$ are bounded above, it follows that the function $F(t, X)$ defined as:

$$F(t, X) = \begin{pmatrix} x \\ y \\ -f(t,x,y,z)z - q(t)g(x,y) - h(x,y,z) \end{pmatrix}$$

is bounded. Since $g(0,0) = 0 = h(0,0,0)$, the only set contained in Ω is the origin. Then by Theorem 14.1 p. 60 – 61 in [11], (5) follows. This completes the proof of Theorem 1.

THEOREM 5. Suppose that $a, a_0, b, b_0, c, q_0, \alpha, \beta, \delta, \mu$ are positive constants and $P_0 \geq 0$ are such that:

(i) hypotheses (i) - (vi) of Theorem 1 hold;

(ii) $|p(t,x,y,z)| \leq P_0 < \infty$.

Then the solution $(x(t), y(t), z(t))$ of (2) is uniform ultimately bounded.

REMARK 6. If $f(t,x,y,z) \equiv f(x,y,z)$ and $q(t) \equiv 1$, then the system (2) reduces to that investigated by Omeike in [7]. Moreover, the condition required here on $f(t,x,y,z)$ to imply that every solution $(x(t), y(t), z(t))$ of (2) to be uniform ultimately bounded is weaker here than that used by Omeike in [7] for the nonlinear third-order differential equation (3), since there it was required that $f(x,y,z) > a$.

LEMMA 7. Subject to the assumptions of Theorem 5, there exists a positive constant $D_2 = D_2(a,b,c,\alpha,\beta,\delta,\mu)$ such that along any solution $(x(t), y(t), z(t))$ of (2) $V' \leq -D_2(x^2+y^2+z^2)$.

PROOF. Along a solution $(x(t), y(t), z(t))$ of (2), we have:

$$V'_{(1.2)} = V'_{(2.1)} + [\beta x + (\alpha + a)y + 2z]p(t,x,y,z).$$

From estimate (14), hypothesis (ii) of Theorem 5 and the Schwartz inequality, we obtain:

$$V'_{(1.2)} \leq -\delta_2(x^2+y^2+z^2) + \delta_3(x^2+y^2+z^2)^{1/2} \quad \textbf{(15)}$$

where $\delta_3 = 3^{1/2} P_0 \max\{\beta; \alpha + a; 2\}$. Choose $(x^2+y^2+z^2)^{1/2} \geq \delta_4 = 2\delta_2^{-1}\delta_3$, inequality (15) becomes $V'_{(1.2)} \leq -\delta_5(x^2+y^2+z^2)$, where $\delta_5 = \frac{1}{2}\delta_2$. This completes the proof of Lemma 7.

PROOF OF THEOREM 5. From conditions (i) and (ii) of Lemma 4, Lemma 7 and Theorem 10.4 in [11] p 42, it follows that the solution $(x(t), y(t), z(t))$ of (2) is uniform ultimately bounded.

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SUGGESTED CITATION

Ademola, A.T. and P.O. Arawomo. 2009. "Boundedness and Stability of Solutions of Some Nonlinear Differential Equations of the Third-Order". *Pacific Journal of Science and Technology.* 10(2):187-193.