The Use of Time as an Optimality Performance Criterion in Manpower Control.

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ABSTRACT

In a graded manpower system, an apparently sensible transition rate may, in due course, show a tendency for certain grades to grow at the expenses of others. Achieving operational objectives in manpower planning is of paramount importance. A typical objective may be to reach a desired structure by a certain time in a changing environment or with the smallest possible cost. Therefore a certain degree of control is sensible at various points in time and for various reasons. In this paper, the concept of time as an optimality performance criterion is used to obtain an optimal recruitment control vector for a manpower system modeled by a stochastic differential equation via the necessary condition of Pontryagin theorem. It is also shown that the optimal recruitment control vector minimizes the control time globally. Condition under which the system is controllable is examined.

(Keywords: time, optimality, recruitment control, performance criterion, stochastic differential equation)

INTRODUCTION

Models developed for manpower systems can be categorized into three main groups, namely: The demand models, supply models, and control models. The demand models are concerned with the prediction of future demand for manpower. This involves looking at productivity changes, technological changes, market forces, trend, and corporate strategies; for example Marshall (1973).

The supply models are concerned with predicting the future supply of manpower. This involves having knowledge of current manpower stock, recruitment and wastage patterns, working conditions, promotion policy, and labor market trends; for example McClean et al. (1992).

The control models on the other hand have two aspects: maintainability (maintaining a given structure) and attainability (feasibility of attaining a desired structure): for example Nwaigwe (2008).

The demand and supply models are transition models and are concerned with the dynamics of the manpower system and the changing trend of stocks and flows. The dynamics of the system have some undesirable consequences on the structure of the system. For instance, in a graded system, an apparently sensible transition rate may, in due course, show a tendency for certain grades to grow at the expenses of others, a transition rate prevailing during a period of economic boom will be too high to maintain during recession, Uche (1984). Therefore, for adequate planning purposes, it is required that the net effect of transitions is zero; a situation we can use the term stationary to describe. This idea plays an important role in manpower planning where objectives can often be stated in terms of achieving a stationary state in which the principal variables have stable and acceptable values. These variables are recruitment, transfers promotions, and retirements. Establishing a stable state as a desirable goal in manpower planning delves naturally into the area of manpower control.

It is evident that 'forces' acting on a manpower system can be divided into two main groups. These are those forces which can be controlled at will by a manpower planner, for example, promotion and recruitment and those which cannot be controlled fully, for example wastage. By judiciously and continually adjusting the controlled variables, we can often get the system to perform in a way consistent with a specific objective. A typical objective may be to reach a desired structure by a certain time in a changing environment or with the smallest possible cost.

Certain degree of control is sensible at various points in time and for various reasons.

The subject of optimal control has attracted the attention of several authors in the mathematical sciences, for example, Washburn (1979), presented a semi-group formulation of boundary input problems for systems governed by parabolic partial differential equation. He established under general condition, a useful bound on the operator kernel of an input map and used the bound to study the input map in a time optimal boundary control problem.

Aubin and Clarke (1979), considered a class of optimal control problems in which the cost functional is locally Lipschitz (not necessarily convex or differentiable) and the dynamics linear and/or convex. By using generalized gradient and duality methods of functional analysis, they obtained necessary conditions in which the dual variables admit interpretation as shadow price or rate of change of the value function.

In the area of manpower control, a couple of authors have discussed the problem of manpower control in different ways. Mehlmann (1980) for example used the concept of dynamic programming to obtain optimal recruitment and transition strategies within a discrete time Markovian framework. The problem of manpower control by departmentalization in an extended Markov framework is extensively discussed in Ossai (2008). Udom and Uche (2008) developed an optimal promotion cost control model for distinguishing different promotion control strategies. This optimal cost model is able to tackle the problem which arose in Uche (1984) in which the solution to a control equation resulted to a set of admissible promotion control strategies.

In most manpower control problems, there may be more than one way of reaching a desired structure or maintaining a given structure. In situations like this, one is faced with the problem of selecting the control strategy that is best in some sense. This is an aspect of optimal manpower control in which, interest is on the problem of compelling a system in this case, a manpower system to behave in some best possible way. Definitely, the exact control strategy depends on the criterion used to decide what is meant by best.

This paper, examines the condition under which a manpower system modeled by a differential equation is controllable and uses time as an optimality performance criterion in controlling the manpower system with quadratic index in both state and control spaces.

MINIMUM TIME OPTIMALITY PERFORMANCE CRITERION

Here the control strategy is to be chosen in such a way as to transfer the system from an initial state n to a desired state n^* in the shortest possible time. This is equivalent to minimizing the performance index,

$$C = \int_{t_0}^{t_1} dt$$

where t_1 is the first instant of time at which the desired state n^* is reached.

Pontryagin Maximum Principle Theorem

Let $u^*(t)$ be an admissible control with corresponding trajectory n^* that transfers a controlled system from n_0 at time t_0 to n_1 at some unspecified time t_1. Then in order that u^* and n^* be optimal (that is minimize some performance index) it is necessary that there exist a non-trivial vector $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ satisfying the Hamiltonian H and co-state equations such that for every t in $t_0 \le t \le t_1$,H attains its maximum with respect to,

u at $u = u^*(t)$

$H(\lambda^*, x^*, u^*) = 0$ *and* $\lambda_0 \le 0$ at $t = t_1$

where λ^* is the solution of the co-state equation for $u = u^*(t)$.

Furthermore, it can be shown that $H(\phi^*(t), x^*(t), u^*[t])$ *and* $\lambda_0(t)$ *are* constants so that the Hamiltonian H= 0 and $\lambda_0 \le 0$ at each point on an optimal trajectory.

Manpower System

Consider a manpower system whose members are divided into k categories. Let $n_i(t)$ denote the number of people in category i at time t (t = 0,1,2,…), N(t) = $\sum_i n_i(t)$ the total number in the system and R(t) the number of recruits to the system at time t. A member of category i moves to category j with probability P_{ij}; where $\sum_j^k P_{ij} < 1.$

Because transition of members out of the system is allowed; we have $P_{i,k+1}$ to be the probability of a member in category i moving out of the system such that $\sum_j P_{ij} + P_{i,k+1} = 1$.

The total number of recruits $R(t)$ is distributed to the k-categories according to the proportion r_{0i} with $\sum_i r_{0i} = 1$ and Q is a $k \times k$ diagonal matrix of $R(t)$.

Using the above notations, the system can be described with the following recursive relation:

$$n_i(t+\partial t) = \sum_i^k P_{ij} n(t) + r_{0j}(t+\partial t)Q$$

or in matrix form

$$n(t+\partial t) = n(t)P + r_{0j}(t+\partial t)Q$$

Differentiating the above equation with respect to t , results to the following stochastic differential equation $\dot{n}(t) = P\,n(t) + Qr(t)$

CONTROLLABILITY OF THE SYSTEM

At this point, we note that an essential first step in dealing with many optimal control problems is to determine whether desired objectives can be achieved by manipulating the chosen control instrument in a certain way. If not then either the objective will have to be modified or control will have to be applied in some different fashion. Here we discuss the general property of being able to transfer a system from any given initial state to a desired state by means of a suitable choice of control function. Particularly, the system $\dot{n}(t) = P\,n(t) + Qr(t)$ is completely controllable if for any t_0, any initial state $\dot{n}(t_0)$ and any desired state $\dot{n}^*(t_1)$ there exists a finite time $t_1 > t_0$ and a control strategy $r(t), t_0 \le t \le t_1$, such that $\dot{n}(t_1) = n^*(t_1)$.

Proposition 1

The system $\dot{n}(t) = P\,n(t) + Qr(t)$ is completely controllable (at t_0) if and only if the $k \times k$ controllability matrix,

$$U = \left[Q, PQ, P^2Q, \ldots\ldots P^{K-1}Q\right]$$

has rank k.

Proof

Necessary Condition: Suppose that the system is completely controllable and wish to prove rank of U=k. This is done by assuming rank of U <k, which leads to a contradiction. For then there would exist a k- row vector $v \neq 0$ such that $vQ = 0, vPQ = 0, \ldots\ldots vP^{k-1}Q = 0.$ Using the solution of the system and the definition of exponential matrix:

$$e^{Pn} = I + Pn + \frac{1}{2!}P^2n^2 + \ldots + \frac{1}{k!}P^kn^k + \ldots \quad \textbf{(1)}$$

We obtain,

$$-\,n(t_0) = \int_{t_0}^{t_1} e^{-Pt} Qr\ (t)\,dt \qquad \textbf{(2)}$$

Using Equation (1) in Equation (2) and multiplying on the left by v gives $vn(t_0) \equiv 0$. Since the system is completely controllable this must hold for any vector $n(t_0)$, which implies that $v = 0$, thus contradicting the assumption that rank of $U < k$.

Sufficiency condition: Now suppose that U is of rank k. We want to show that the system is completely controllable. Consider the case $n(t_0)=0$. If the system were not controllable, then there would be a nonzero vector v which could not be reached via any control at t=1, that is

$$v\int_0^1 e^{P(1-s)}Qr(s)ds=0$$

or for piecewise continuous control

$$v'e^{(1-s)}Q=0 \quad (0\le s\le 1) \qquad \textbf{(3)}$$

By a DuBois-Reymond type argument. For s = 1, we would have $v'Q=0$. Upon repeated integration of Equation (3) with respect to s and setting s = 1, we would also obtain $vQ=0, vPQ=0, vP^2Q.....vP^{k-1}Q=0.$ This contradicts the assumption that U is of rank k. Hence the system is completely controllable.

OPTIMAL TIME RECRUITMENT CONTROL MODEL

Consider a manpower system represented by the following stochastic differential equation:

$$\dot{n}(t)=P\,n(t)+Qr(t) \qquad \textbf{(4)}$$

where $\dot{n}(t)$ is an infinitesimal change in the manpower structure $n(t)$ in a small interval of time, $Pn(t)$ is an infinitesimal change in the manpower structure $n(t)$ resulting from promotions in a small interval of time and $Qr(t)$ is an infinitesimal change in the manpower structure $n(t)$ resulting from recruitment in a small interval of time.

The system represented by Equation (4) is to be controlled by recruitment during the fixed interval $t_0\le t\le t_1$ from an initial state $n(t_0)$ to a desired state $n(t_1)$ in such a way that the cost function, C which is quadratic in state and control spaces,

$$C=\frac{1}{2}n'(t_1)Sn'(t_1)+\frac{1}{2}\int_{t_0}^{t_1}\left[n'V(t)n+r'W(t)r\right]dt \qquad \textbf{(5)}$$

is minimized, subject to the equation of state and the initial condition $n(t_0)=n_0$ where S,V and W are $k\times k$ diagonal matrices with ith diagonal elements being weights reflecting the relative importance attached to the ith grade size and the matrices are restricted to be positive definite.
The Hamiltonian for this problem is:

$$H=\lambda_0\left(v_{ij}n_in_j+w_{ij}r_ir_j\right)+\lambda_i\left(p_{ij}n_j+q_{ij}r_j\right) \qquad \textbf{(6)}$$

Where,

$$\lambda\, satisfy\;\; \dot{\lambda}_i=\frac{-\partial H}{\partial n_i}=-v_{ij}n_j-\lambda_i p_{ij} \qquad \textbf{(7)}$$

To minimize C, according to Pontryangin theorem, it is equivalent to maximizing H and it is necessary for:

$$\frac{\partial H}{\partial r_i}=w_{ij}r_i+\lambda_i q_i=0 \qquad \textbf{(8a)}$$

$$and\;\; \lambda(t_1)=Sn(t_1) \qquad \textbf{(8b)}$$

To maximize H we require $r=r^*$
from equation (4) we have that,

$$Wr^*+\lambda_i Q=0 \qquad \textbf{(9)}$$

provided W is invertible we can write Equation (9) as:

$$r^*=-W^{-1}Q'\lambda \qquad \textbf{(10)}$$

we can now use Equation (10) to eliminate r from equation (4) to obtain:

$$\dot{n}=Pn-QW^{-1}Q'\lambda \qquad \textbf{(11)}$$

substituting Equation (11) into the state and co-state equations we have:

$$Z=\begin{pmatrix}\dot{n}\\ \dot{\lambda}\end{pmatrix}=\begin{pmatrix}P & -QW^{1}Q'\\ -V & -P'\end{pmatrix}\begin{pmatrix}n\\ \lambda\end{pmatrix} \qquad \textbf{(12)}$$

Let $Z(t,t_0)$ be the resolvent matrix of Equation (12) hence:

$$Z(t) = Z(t,t_0)\begin{pmatrix} n(t_0) \\ \lambda(t_0) \end{pmatrix}$$

The problem now is to solve Z for the boundary value at t_1. By partitioning,

$$Z(t,t_0) = \begin{pmatrix} Z_{11}(t,t_0) & Z_{12}(t,t_0) \\ Z_{21}(t,t_0) & Z(t,t_0) \end{pmatrix},$$

we can write Equation (11) as:

$$\begin{aligned} n(t_1) &= Z_{11}(t_1,t)n(t) + Z_{12}(t_1,t)\lambda(t) \\ Sn(t_1) &= Z_{12}(t_1,t)n(t) + Z_{22}(t_1,t)\lambda(t) \end{aligned} \quad \textbf{(13)}$$

Solving Equation (13) for $n(t_1)$ and $\lambda(t)$ we have $\lambda(t) = \Lambda(t,t_1)n(t)$ and $n(t_1) = N(t_1,t)n(t)$

where

$$\Lambda(t,t_1) = [Z_{22}(t_1,t) - SZ_{12}(t,t_1)]^{-1}[sZ_{11}(t_1,t) - Z_{21}(t_1t)],$$

$$N(t_1,t) = \left[I_n - Z_{12}(t_1,t)Z_{22}^{-1}(t_1,t)S\right]^{-1}\left[Z_{11}(t_1,t) - Z_{12}(t_1,t)Z_{22}^{-1}(t_1,t)Z_{21}(t_1,t)S\right]$$

Therefore equation (11) becomes,

$$\begin{aligned} r^* &= -W^{-1}Q'\Lambda(t,t_1)n(t) \\ &= -W^{-1}Q'\left[-P - SQW^{-1}Q'\right]^{-1}[SP+V]n^* \end{aligned} \quad \textbf{(14)}$$

The above equation is the optimal recruitment control vector.

Proposition 2

The recruitment control vector r^* defined by Equation (14) minimizes the control time globally.

Proof

The function $F = \frac{1}{2}(n'Vn + r'Wr)$ is convex in (n,r) because both V and W have been restricted to be positive definite. Let $\tilde{r}$ be some other recruitment control vector satisfying the optimality conditions of the Pontryagin theorem, then we have

$$2[F(t,\tilde{n},\tilde{r}) - F(t,n,r)] \geq (\tilde{n} - n)' V(\tilde{n} - n) + (\tilde{r} - r*)' W(\tilde{r} - r^*)$$

similarly, we have for some positive definite matrix S a similar inequality hold, that is, $S\tilde{n}(t_1) - Sn(t_1) \geq (\tilde{n}(t_1) - n(t_1))S(n^*(t_1) - n(t_1))$ with these two inequalities, the difference between $C(\tilde{n},\tilde{r})$ and $C(n^*,r^*)$ satisfy,

$$\tilde{C} - C^* \geq \Delta n'(t_1)S\Delta n(t_1) + \int_{t_0}^{t_1}[\Delta n'V\Delta n + \Delta r'W\Delta r]dt \quad \textbf{(15)}$$

where $\Delta g = \tilde{g} - g^*$.

Now, using Equations (7) and (10) in Equation (15), and integrating by parts, we obtain the following result:

$$\tilde{C} - C^* \geq \Delta n'(t_1)[Sn(t_1) - \lambda(t_1)] + \Delta n'(t_0)\lambda(t_0) + \int_{t_0}^{t_1}\left[(\Delta n')^{\cdot} - \Delta n'P - \Delta rQ\right]\lambda dt$$

The first and second terms of the above equation vanish because $\lambda(t_1) = Sn(t_1)$ in Equation (8b). For similar reason, the third term vanishes, thus we have,

$\tilde{C} - C^* \geq 0$, as required.

ILLUSTRATIVE EXAMPLE

Consider a five grade hierarchical manpower system which assumes the following parameter values:

$n(t_0) = (300 \quad 225 \quad 150 \quad 100 \quad 68)$,

the given structure:

$$P = \begin{bmatrix} 0.60 & 0.20 & 0 & 0 & 0 \\ 0 & 0.80 & 0.15 & 0 & 0 \\ 0 & 0 & 0.75 & 0.21 & 0 \\ 0 & 0 & 0 & 0.85 & 0 \\ 0 & 0 & 0 & 0 & 0.95 \end{bmatrix} \quad \textit{promotion matrix}$$

$R(t) = 65$ for all t; the number of recruits $n(t_1) = (165 \quad 150 \quad 160 \quad 150 \quad 180)$, the desired structure.

The weighting matrices S,W,V are assume to be identity matrices which means that all the grade sizes have equal weight.

The promotion matrix shows a system in which promotion prospects diminish as one progresses up the hierarchy. Wastage rates fall from 0.2 in the lowest grade to 0.04 at the top. Recruitment is fixed at 65 recruits per unit time, so as to keep the total size roughly constant. The problem is first to check if the system represented above is controllable. This is done by examining the rank of the controllability matrix. If the system is controllable, then the second problem is to obtain a recruitment vector that can move the system from $n(t_0)$ to $n(t_1)$ in such a way that a performance index of the form:

$$J = \int_{t_0}^{t_1} dt$$

is minimized (in the shortest possible time).

The results are presented below:

The contrability matrix is Columns 1 through 7

65.0000	0	0	0	0	39.0000	13.0000
0	65.0000	0	0	0	0	52.0000
0	0	65.0000	0	0	0	0
0	0	0	65.0000	0	0	0
0	0	0	0	65.0000	0	0

Columns 8 through 14

0	0	0	23.4000	18.2000	1.9500	0
9.7500	0	0	0	41.6000	15.1125	2.0475
48.7500	13.6500	0	0	0	36.5625	21.8400
0	55.2500	6.5000	0	0	0	46.9625
0	0	62.4000	0	0	0	0

Columns 15 through 21

0	14.0400	19.2400	4.1925	0.4095	0	8.4240
0	0	33.2800	17.5744	4.9140	0.2047	0
1.3650	0	0	27.4219	26.2421	3.4944	0
11.7650	0	0	0	39.9181	15.9906	0
59.9040	0	0	0	0	57.5078	0

Columns 22 through 25

18.2000	6.0304	1.2285	0.0410
26.6240	18.1728	7.8675	0.6880
0	20.5664	28.0644	5.9788
0	0	33.9304	19.3428
0	0	0	55.2075

The rank of the controllability matrix is 5.Hence the system is controllable.

$$r^* = -WQ'[-P' - SQW^{-1}Q'][SP + V]n(t_1)$$
$$= (0.18 \quad 0.21 \quad 0.22 \quad 0.20 \quad 0.19)$$

CONCLUSION

In this paper, a manpower system is modeled as a stochastic differential equation. The concept of time as an optimality performance criterion is used to obtain an optimal recruitment control vector for the manpower system via the necessary condition of Pontryagin theorem. It is also shown that the optimal recruitment control vector minimizes the control time globally. The condition under which the system is controllable is also examined.

In the illustrative example, the system specified is controllable and the recruitment vector that minimizes the control time is:

$$r^* = (0.18 \quad 0.21 \quad 0.22 \quad 0.20 \quad 0.19).$$

This means that if the recruits are distributed in the system in accordance with this recruitment vector, the desired structure would be achieved in the shortest possible time.

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